

An Estimate on the Number of Eigenvalues of a Quasiperiodic Jacobi Matrix of Size n Contained in an Interval of Size n^{-C}

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Abstract

We consider infinite quasi-periodic Jacobi self-adjoint matrices for which the three main diagonals are given via values of real analytic functions on the trajectory of the shift $x \rightarrow x + \omega$. We assume that the Lyapunov exponent $L(E_0)$ of the corresponding Jacobi cocycle satisfies $L(E_0) \geq \gamma > 0$. In this setting we prove that the number of eigenvalues $E_j^{(n)}(x)$ of a submatrix of size n contained in an interval I centered at E_0 with $|I| = n^{-C_1}$ does not exceed $(\log n)^{C_0}$ for any x . Here $n \geq n_0$, and n_0, C_0, C_1 are constants depending on γ (and the other parameters of the problem).

1 Introduction

Denote $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and let $a : \mathbb{T} \rightarrow \mathbb{R}$, and $b : \mathbb{T} \rightarrow \mathbb{C}$ be real analytic functions, with b not identically zero. Let $\omega \in (0, 1)$ satisfy a (generic) Diophantine condition of the form

$$\|n\omega\| \geq \frac{C_\omega}{n(\log n)^\alpha},$$

where $\alpha > 1$ is fixed. We consider the quasiperiodic Jacobi operator $H(x, \omega)$ defined on $l^2(\mathbb{Z})$ by

$$[H(x, \omega)\phi](k) = -b(x + (k+1)\omega)\phi(k+1) - \overline{b(x + k\omega)}\phi(k-1) + a(x + k\omega)\phi(k).$$

The important special case given by $b \equiv 1$ (Schrödinger operator) has been studied extensively (see the monograph [Bou05]). The study of results that apply to quasiperiodic Jacobi operators in such a general setting has been launched by the recent work of Jitomirskaya, Koslover, and Schulteis [JKS09] and Jitomirskaya and Marx [JM11]. In particular, they studied the extended Harper's model which corresponds to $a(x) = 2\cos(2\pi x)$, $b(x) = \lambda_1 e^{2\pi i(x-\omega/2)} + \lambda_2 + \lambda_3 e^{-2\pi i(x-\omega/2)}$ (see [JKS05, JM10]). Further motivation for the study of these operators comes from

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the general fact that quasiperiodic Jacobi operators are necessary for the solution of the inverse spectral problem for discrete quasiperiodic operators of second order, and for the solution of the Todda Lattice with quasiperiodic initial data.

The main objective of this work is to estimate the number of Dirichlet eigenvalues of the problem on a finite interval of length n which fall into a given interval of size n^{-C} . This type of estimate plays a central role in the work of Goldstein and Schlag [GS01, GS08]. In our analysis we use many ideas and methods of their work. On the other hand, as it was noted in [JM11], the singularities (associated with the zeros of b) of the corresponding matrix-functions introduce considerable technical difficulties. These difficulties are addressed by using a large deviation theorem for subharmonic functions ([GS01, Theorem 3.8]) applied to $\log |b|$, which will allow us to include the singularities in the exceptional sets. The derivation of the large deviation estimate for the characteristic polynomials via the method of [GS08] becomes especially complicated, even if b would have no zeros. We show how to get around these difficulties by introducing a different derivation which makes a finer use of the cocycle structure (see the proof of Lemma 4.2). Our estimate on the number of eigenvalues also improves on the estimate in [GS08].

The methods we will employ are complex analytic, so from now we canonically identify \mathbb{T} with the unit circle in \mathbb{C} . It is known that a and b can be extended to be (complex) analytic on a neighborhood of \mathbb{T} . Let $\tilde{b}(z) := \overline{b(1/\bar{z})}$ denote the analytic extension of \tilde{b} . We now extend the definition of $H(\cdot, \omega)$, to a neighborhood on which both a and b can be extended, by

$$[H(z, \omega)\phi](k) = -b(z + (k+1)\omega)\phi(k+1) - \tilde{b}(z + k\omega)\phi(k-1) + a(z + k\omega)\phi(k).$$

Note that $H(\cdot, \omega)$ is not necessarily self-adjoint off \mathbb{T} . For simplicity we make the notational convention that $z + k\omega := z \exp(2\pi i k \omega)$, for $z \in \mathbb{C}$ and $k \in \mathbb{Z}$.

We consider the finite Jacobi submatrix on $[0, n-1]$, denoted by $H^{(n)}(z, \omega)$, and defined by

$$\begin{bmatrix} a(z) & -b(z+\omega) & 0 & \dots & 0 \\ -\tilde{b}(z+\omega) & a(z+\omega) & -b(z+2\omega) & \dots & 0 \\ \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & -\tilde{b}(z+(n-1)\omega) & a(z+(n-1)\omega) \end{bmatrix}.$$

Let $L(E)$ be the Lyapunov exponent associated with $H(x, \omega)$ (see (2.11)). Our main result is as follows.

Main Theorem. *Assume that $E_0 \in \mathbb{R}$ is such that $L(E_0) \geq \gamma > 0$. Then there exist constants $C_0 = C_0(\omega)$, $C_1 = C_1(a, b, E_0, \omega, \gamma)$, and $n_0 = n_0(a, b, E_0, \omega, \gamma)$ such that for every $x \in \mathbb{T}$ and $n \geq n_0$ the number of eigenvalues for $H^{(n)}(x, \omega)$ located in $\{E : |E - E_0| < n^{-C_1}\}$ is at most $(\log n)^{C_0}$ and furthermore, for any $x_0 \in \mathbb{T}$ and $n \geq n_0$ the number of zeros for $\det(H^{(n)}(\cdot, \omega) - E_0)$ contained in $\{z : |z - x_0| < n^{-1}\}$ is at most $(\log n)^{C_0}$.*

In the Schrödinger case such estimates and further refinements were obtained by Goldstein and Schlag (see [GS08, Proposition 4.9]). In fact we will prove a slightly stronger theorem, Theorem 4.13.

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2 Preliminaries

We proceed by introducing some notation and giving an overview of the methods. For ϕ satisfying the difference equation $H(z, \omega)\phi = E\phi$ let M_n be the matrix such that

$$\begin{bmatrix} \phi(n) \\ \phi(n-1) \end{bmatrix} = M_n \begin{bmatrix} \phi(0) \\ \phi(-1) \end{bmatrix}, \quad n \geq 1.$$

We call M_n the fundamental matrix. We clearly have

$$M_n(z) = \prod_{j=n-1}^0 \left(\frac{1}{b(z+(j+1)\omega)} \begin{bmatrix} a(z+j\omega) - E & -\tilde{b}(z+j\omega) \\ b(z+(j+1)\omega) & 0 \end{bmatrix} \right),$$

for z such that $\prod_{j=1}^n b(z+j\omega) \neq 0$. Note that in order to simplify the notation we suppressed the dependence on ω and E . We will be doing this throughout the paper whenever possible. From now on, if needed, we will include the set on which the matrices M_n are not defined in the exceptional sets.

It is straightforward to see that

$$M_n(z) = \begin{bmatrix} f_n(z) & -\frac{\tilde{b}(z)}{b(z+\omega)} f_{n-1}(z+\omega) \\ f_{n-1}(z) & -\frac{\tilde{b}(z)}{b(z+\omega)} f_{n-2}(z+\omega) \end{bmatrix}, \quad (2.1)$$

with

$$f_n(z) = \frac{1}{\prod_{j=1}^n b(z+j\omega)} f_n^a(z), \quad (2.2)$$

where

$$f_n^a(z) = \det [H^n(z, \omega) - E].$$

Since $f_n^a(x, E)$ is the characteristic polynomial of $H^{(n)}(x, \omega)$ so it is natural to estimate the number of eigenvalues by applying Jensen's formula to f_n^a . For this to work we need upper and lower estimates on $\log |f_n^a|$. These estimates will follow from the deviations estimates for the fundamental matrix and its entries (see Theorem 3.10 and Proposition 4.10).

The main tools for obtaining the deviations estimates for the fundamental matrix are a deviations estimate for subharmonic functions and the Avalanche Principle, both of which we recall next. In what follows \mathcal{A}_ρ will denote the annulus $\{z \in \mathbb{C} : |z| \in (1-\rho, 1+\rho)\}$ and we fix $p > \alpha + 2$.

Theorem 2.1. ([GS01, Theorem 3.8]) Let u be a subharmonic function and let

$$u(z) = \int_{\mathbb{C}} \log |z - \zeta| d\mu(\zeta) + h(z)$$

be its Riesz representation on a neighborhood of \mathcal{A}_ρ . If $\mu(\mathcal{A}_\rho) + \|h\|_{L^\infty(\mathcal{A}_\rho)} \leq M$ then for any $\delta > 0$ and any positive integer n we have

$$\text{mes} \left(\left\{ x \in \mathbb{T} : \left| \sum_{k=1}^n u(x + k\omega) - n \langle u \rangle \right| > \delta n \right\} \right) < \exp(-c_0 \delta n + r_n)$$

where $c_0 = c_0(\omega, M, \rho)$ and

$$r_n = \begin{cases} C_0 (\log n)^p & , n > 1 \\ C_0 & , n = 1, \end{cases}$$

with $C_0 = C_0(\omega, p)$.

If p_s/q_s is a convergent of ω and $n = q_s > 1$ then one can choose $r_n = C_0 \log n$. One can keep this in mind, but we will make no use of it.

Proposition 2.2. (Avalanche Principle; [GS08, Proposition 3.3]) Let A_1, \dots, A_n , $n \geq 2$, be a sequence of 2×2 matrices. If

$$\max_{1 \leq j \leq n} |\det A_j| \leq 1,$$

$$\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n,$$

and

$$\max_{1 \leq j < n} (\log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\|) < \frac{1}{2} \log \mu$$

then

$$\left| \log \|A_n \dots A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| < C_0 \frac{n}{\mu}$$

with some absolute constant C_0 .

In [GS01] (where $b \equiv 1$) one takes advantage of the fact that $\log \|M_n(\cdot)\|$ is subharmonic (on a neighborhood of \mathbb{T}) and that it is almost invariant to get a first deviations estimate by using Theorem 2.1. Next, this estimate is used to apply the Avalanche Principle, which together with the almost invariance yields a sharper deviations estimate. Almost invariance refers to the fact that

$$\left| \log \|M_n(x)\| - \frac{1}{l} \sum_{k=0}^{l-1} \log \|M_n(x + k\omega)\| \right| \leq Cl, \quad x \in \mathbb{T}.$$

In our case $\log \|M_n(\cdot)\|$ is not necessarily subharmonic, the Avalanche Principle (as stated) cannot be applied to M_n , because it is possible that $|\det M_n| \not\leq 1$, and

the almost invariance may fail to hold on \mathbb{T} . To work around these issues it is natural to use the following two matrices associated with M_n :

$$M_n^a(z) = \left(\prod_{j=1}^n b(z + j\omega) \right) M_n(z) \quad (2.3)$$

and

$$M_n^u(z) = \frac{1}{\sqrt{|\det M_n(z)|}} M_n(z) = \left(\prod_{j=0}^{n-1} \left| \frac{b(z + (j+1)\omega)}{\tilde{b}(z + j\omega)} \right|^{1/2} \right) M_n(z) \quad (2.4)$$

$M_n^a(\cdot)$ is analytic and hence $\log \|M_n^a(\cdot)\|$ is subharmonic, and $M_n^u(\cdot)$ is unimodular (i.e. $|\det M_n^u| = 1$). Clearly, we will apply Theorem 2.1 to $\log \|M_n^a\|$ and the Avalanche Principle to M_n^u . Note that $\log \|M_n^a(\cdot)\|$ would be subharmonic even if we had \bar{b} instead of \tilde{b} , however \tilde{b} is needed to ensure that f_n^a is analytic. Furthermore, if we have \bar{b} instead of \tilde{b} the function $\log |f_n^a(\cdot)|$ is not necessarily subharmonic.

Using (2.1), (2.3), and (2.4) it is straightforward to check that

$$M_n^a(z) = \begin{bmatrix} f_n^a(z) & -\tilde{b}(z) f_{n-1}^a(z + \omega) \\ b(z + n\omega) f_{n-1}^a(z) & -\tilde{b}(z) b(z + n\omega) f_{n-2}^a(z + \omega) \end{bmatrix}, \quad (2.5)$$

$$M_n^u(z) = \begin{bmatrix} f_n^u(z) & -\frac{\tilde{b}(z)}{b(z+\omega)} \left| \frac{b(z+\omega)}{\tilde{b}(z)} \right|^{1/2} f_{n-1}^u(z + \omega) \\ \left| \frac{b(z+n\omega)}{\tilde{b}(z+(n-1)\omega)} \right|^{1/2} f_{n-1}^u(z) & -\frac{\tilde{b}(z)}{b(z+\omega)} \left| \frac{b(z+n\omega)b(z+\omega)}{\tilde{b}(z+(n-1)\omega)\tilde{b}(z)} \right|^{1/2} f_{n-2}^u(z + \omega) \end{bmatrix}, \quad (2.6)$$

where

$$f_n^u(z) = \left(\prod_{j=0}^{n-1} \left| \frac{b(z + (j+1)\omega)}{\tilde{b}(z + j\omega)} \right|^{1/2} \right) f_n(z) \quad (2.7)$$

(f_n and f_n^a have already been defined).

Let $S_n(z) = \sum_{k=0}^{n-1} \log |b(z + k\omega)|$ and $\tilde{S}_n(z) = \sum_{k=0}^{n-1} \log |\tilde{b}(z + k\omega)|$. From (2.3) and (2.4) we get

$$\log \|M_n(z)\| = -S_n(z + \omega) + \log \|M_n^a(z)\|, \quad (2.8)$$

and

$$\log \|M_n^u(z)\| = -\frac{1}{2} (\tilde{S}_n(z) + S_n(z + \omega)) + \log \|M_n^a(z)\|. \quad (2.9)$$

It will be easy to see that these relations together with Theorem 2.1 applied to $\log |b|$ and $\log |\tilde{b}|$ allow us to pass from deviations estimates for M_n^a to deviations estimates for M_n and M_n^u (see for example Corollary 3.6).

Even though we will apply the Avalanche Principle to M_n^u the conclusion will also hold for M_n^a and M_n . We will make this more precise. Let $n = \sum_{j=1}^m l_j$, $s_k = \sum_{j=1}^k l_j$ where m, l_1, \dots, l_m are positive integers. We assume that $s_0 = 0$.

By saying that, for example, the conclusion of the Avalanche Principle applied to M_n^u also holds for M_n^a we mean that

$$\begin{aligned} \log \|M_n^u(z)\| + \sum_{j=2}^{m-1} \log \left\| M_{l_j}^u(z + s_{j-1}\omega) \right\| \\ - \sum_{j=1}^{m-1} \log \left\| M_{l_{j+1}}^u(z + s_j\omega) M_{l_j}^u(z + s_{j-1}\omega) \right\| = \log \|M_n^a(z)\| \\ + \sum_{j=2}^{m-1} \log \left\| M_{l_j}^a(z + s_{j-1}\omega) \right\| - \sum_{j=1}^{m-1} \log \left\| M_{l_{j+1}}^a(z + s_j\omega) M_{l_j}^a(z + s_{j-1}\omega) \right\|. \end{aligned}$$

This follows easily from (2.9).

The deviations estimate for $\log |f_n^a|$ is just the John-Nirenberg inequality. The needed *BMO* norm bound will be obtained by using the “*BMO* splitting lemma” [BGS01, Lemma 2.3]. As in the case for the fundamental matrix, we first obtain a rough estimate (Lemma 4.9) that allows us to apply the Avalanche Principle in order to obtain a better estimate. We follow the approach from [GS08] with the notable exception of the proof of Lemma 4.2 (cf. [GS08, Lemma 2.7]). This is the only place where the difficulties come not only from the possible zeroes of b but also from the fact that b is not constant.

We will obtain a uniform upper bound for $\log |f_n^a(\cdot)|$ on \mathbb{T} from an uniform upper bound for $\log \|M_n^a(\cdot)\|$ (Proposition 3.14) and the obvious inequality $\log |f_n^a(\cdot)| \leq \log \|M_n^a(\cdot)\|$. The proof of Proposition 3.14 requires that the deviations estimate for $\log \|M_n^a\|$ holds on $r\mathbb{T}$ for r in a neighborhood of 1. Of course this implies that all the results leading to the deviations estimate should also hold on $r\mathbb{T}$. For simplicity we will prove these estimates on \mathbb{T} , however the proofs will be such that the generalization from \mathbb{T} to $r\mathbb{T}$ is immediate. To this end the derivations up to Proposition 3.14 won’t use the fact that $\tilde{b} = \bar{b}$ on \mathbb{T} . However, after that point we only need the results to hold on \mathbb{T} and we will make use of said fact to simplify notation.

The deviations estimates will rely on the positivity of the Lyapunov exponent. Let

$$L_n(r) = \frac{1}{n} \int_{\mathbb{T}} \log \|M_n(rx)\| dx,$$

$$L_n^u(r) = \frac{1}{n} \int_{\mathbb{T}} \log \|M_n^u(rx)\| dx,$$

$$L_n^a(r) = \frac{1}{n} \int_{\mathbb{T}} \log \|M_n^a(rx)\| dx,$$

$$D(r) = \int_{\mathbb{T}} \log |b(rx)| dx,$$

and

$$\tilde{D}(r) = \int_{\mathbb{T}} \log |\tilde{b}(rx)| dx.$$

When $r = 1$ we will omit the r argument, so for example we will write L_n instead of $L_n(1)$. The quantities $L_n^a(r)$, $D(r)$, and $\tilde{D}(r)$ are finite because the integrands are subharmonic (and not identically $-\infty$), and $L_n(r)$ is finite because from (2.8) we have

$$L_n(r) = -D(r) + L_n^a(r). \quad (2.10)$$

By Kingman's subadditive ergodic theorem the following limits exist:

$$L(r) = \lim_{n \rightarrow \infty} L_n(r) = \inf_{n \geq 1} L_n(r), \quad (2.11)$$

$$L^u(r) = \lim_{n \rightarrow \infty} L_n^u(r) = \inf_{n \geq 1} L_n^u(r), \quad (2.12)$$

$$L^a(r) = \lim_{n \rightarrow \infty} L_n^a(r) = \inf_{n \geq 1} L_n^a(r). \quad (2.13)$$

$L = L(E, \omega)$ is called the Lyapunov exponent. From (2.4) it can be seen that

$$L^u(r) = \frac{1}{2}(\tilde{D}(r) - D(r)) + L(r)$$

and in particular, since $D = \tilde{D}$, we have $L = L^u$. Since M_n^u is unimodular it follows that $L_n^u(r) \geq 0$, and hence $L^u(r) \geq 0$. In particular we have that $L = L^u \geq 0$.

Fix $\gamma > 0$. From now on we assume that $L \geq \gamma > 0$. This assumption is needed to apply the Avalanche Principle, so in fact we will use $L^u = L \geq \gamma > 0$. For the results to hold on $r\mathbb{T}$, $r \neq 1$, we will need that r is close enough to 1 so that $L^u(r) \geq \gamma/2 > 0$. Note that the results up to Lemma 3.9 don't use the Avalanche Principle and so they hold without the assumption that $L \geq \gamma > 0$.

Henceforth we will assume that a and b are analytic on the closure of $\mathcal{A}_{\rho_0''}$ with $\rho_0'' > 0$ fixed. We also fix ρ_0 and ρ_0' such that $0 < \rho_0 < \rho_0' < \rho_0''$. The reason for this setup is that $\log \|M_n^a(\cdot)\|$ will have a Riesz representation on $\mathcal{A}_{\rho_0'}$ but we will be able to get the estimates on the Riesz representation (needed for Theorem 2.1) only on \mathcal{A}_{ρ_0} . The estimates before Proposition 3.14 will hold on $r\mathbb{T}$ for every $r \in (1 - \rho_0/2, 1 + \rho_0/2)$ (provided $L^u(r) > 0$) and the constants can be chosen uniformly for all such r . Proposition 3.14 will hold on $r\mathbb{T}$ for every $r \in (1 - \rho_0/4, 1 + \rho_0/4)$ (provided $L^u(r) > 0$).

3 Estimates for the Fundamental Matrix

First we prove the almost invariance of M_n^a (see (3.7)). The following lemma and its corollaries contain the main estimates that are needed to deal with the fact that b could have zeros. If b doesn't have any zeros then all the estimates hold trivially without exceptional sets and everything goes as in [GS01].

In what follows we will keep track of the dependence of the various constants on the parameters of our problem. The dependence on ω will only come up through Theorem 2.1. In order to simplify the notation we won't record the dependence on ρ_0 , ρ_0' , and ρ_0'' (except in the lemmas where ρ_0 appears in the statement). Dependence on any other quantities is such that if the quantity takes values in a compact set, then the constant can be chosen uniformly with respect to that

quantity. The main dependence we are interested in, is that on $|E|$. We denote by $\|\cdot\|_\infty$ the L^∞ norm on $\mathcal{A}_{\rho_0''}$ and we let $\|b\|_* = \|b\|_\infty + \sup_{r \in (1-\rho_0, 1+\rho_0)} |D(r)|$. Note that $\|b\|_* = \|\tilde{b}\|_*$.

Lemma 3.1. *There exist constants $\lambda_0 = \lambda_0(\|a\|_\infty, \|b\|_*, |E|, \omega)$ and $c_0 = c_0(\|b\|_*, \omega)$ such that the following inequalities hold for any positive integer l and any $\lambda \geq \lambda_0$ up to a set (independent of E) of measure less than $\exp(-c_0\lambda l)$:*

$$|\log \|M_l^a(x)\|| \leq \lambda l \quad (3.1)$$

$$\left| \log \left\| M_l^a(x)^{-1} \right\| \right| \leq \lambda l. \quad (3.2)$$

Proof. There exists a constant $C = C(\|a\|_\infty, \|b\|_\infty, |E|) > 0$ such that

$$\log \|M_l^a(x)\| \leq Cl$$

for all x . On the other hand

$$\|M_l^a(x)\| \geq |\det M_l^a(x)|^{1/2} = \prod_{j=0}^{l-1} |\tilde{b}(x+j\omega) b(x+(j+1)\omega)|^{1/2}$$

for all x . Hence

$$\frac{\tilde{S}_l(x) + S_l(x+\omega)}{2} \leq \log \|M_l^a(x)\| \leq Cl \quad (3.3)$$

for all x . From Theorem 2.1 we can conclude that for any $\lambda' > 0$ we have

$$-2\lambda'l \leq \left(\frac{\tilde{D} + D}{2} - \lambda' \right) l \leq \log \|M_l^a(x)\| \leq Cl \leq 2\lambda'l$$

up to a set not exceeding $2 \exp(-c\lambda'l + r_l)$ in measure, provided

$$\lambda' \geq \max \{ -(\tilde{D} + D), C \} / 2.$$

By setting $\lambda = 2\lambda'$ and choosing $\lambda_0 \geq \max \{ -(\tilde{D} + D), C \}$ we have that (3.1) holds up to a set of measure not exceeding $2 \exp(-c\lambda l + r_l)$. Finally, it is easy to see that by choosing λ_0 such that

$$\lambda_0 \geq \frac{2}{c} \sup_{l \geq 1} \frac{\log 2 + r_l}{l}$$

we have

$$2 \exp(-c\lambda l + r_l) \leq \exp\left(-\frac{c}{2}\lambda l\right), \lambda \geq \lambda_0, l \geq 1.$$

This concludes the proof of (3.1).

Since for almost every x we have

$$\begin{aligned} & \begin{bmatrix} a(x+j\omega) - E & -\tilde{b}(x+j\omega) \\ b(x+(j+1)\omega) & 0 \end{bmatrix}^{-1} = \\ & \frac{1}{\tilde{b}(x+j\omega) b(x+(j+1)\omega)} \begin{bmatrix} 0 & \tilde{b}(x+j\omega) \\ -b(x+(j+1)\omega) & a(x+j\omega) - E \end{bmatrix} \end{aligned}$$

it is straightforward to see that there exists a constant $C = C(\|a\|_\infty, \|b\|_\infty, |E|)$ such that

$$-\frac{\tilde{S}_l(x) + S_l(x + \omega)}{2} \leq \log \left\| M_l^a(x)^{-1} \right\| \leq Cl - \tilde{S}_l(x) - S_l(x + \omega)$$

for almost every x . Now (3.2) follows in the same way as (3.1). Note that the exceptional set comes from $\tilde{S}_l(x) + S_l(x + \omega)$ and is thus independent of E . \square

The same type of estimates can be obtained now for M_n and M_n^u . We just record one of the estimates that will be needed later.

Corollary 3.2. *There exist constants $\lambda_0 = \lambda_0(\|a\|_\infty, \|b\|_*, |E|, \omega)$ and $c_0 = c_0(\|b\|_*, \omega)$ such that*

$$\left| \log \left\| M_l^u(x)^{-1} \right\| \right| \leq \lambda l$$

holds for any positive integer l and any $\lambda \geq \lambda_0$ up to a set of measure less than $\exp(-c_0\lambda l)$.

Proof. From (2.4) we have

$$\log \left\| M_l^u(x)^{-1} \right\| = \frac{1}{2} (\tilde{S}_l(x) + S_l(x + \omega)) + \log \left\| M_l^a(x)^{-1} \right\|.$$

Using Theorem 2.1 and (3.2) we get

$$-3\lambda' l \leq \left(\frac{\tilde{D} + D}{2} - 2\lambda' \right) l \leq \log \left\| M_l^u(x)^{-1} \right\| \leq \left(\frac{\tilde{D} + D}{2} + 2\lambda' \right) l \leq 3\lambda' l$$

up to a set of measure less than $2 \exp(-c_1\lambda' l + r_l) + \exp(-c_2\lambda' l) \leq \exp(-c_3\lambda' l)$ provided λ' is large enough. Now we can take $\lambda = 3\lambda'$. \square

Corollary 3.3. *There exist constants $\lambda_0 = \lambda_0(\|a\|_\infty, \|b\|_*, |E|, \omega)$ and $c_0 = c_0(\|b\|_*, \omega)$ such that the following inequalities hold for any positive integers l and n , and any $\lambda \geq \lambda_0$ up to a set (depending on n) of measure less than $\exp(-c_0\lambda l)$:*

$$|\log \|M_l^a(x)\| - lL_l^a| \leq \lambda l \quad (3.4)$$

$$|\log \|M_{n+l}^a(x)\| - \log \|M_n^a(x)\|| \leq \lambda l \quad (3.5)$$

$$|\log \|M_n^a(x + l\omega)\| - \log \|M_n^a(x)\|| \leq \lambda l \quad (3.6)$$

$$\left| \log \|M_n^a(x)\| - \frac{1}{l} \sum_{k=0}^{l-1} \log \|M_n^a(x + k\omega)\| \right| \leq \lambda l. \quad (3.7)$$

Proof. By integrating (3.3) we get

$$\frac{\tilde{D} + D}{2} \leq L_l^a \leq C. \quad (3.8)$$

This and (3.1) imply (3.4).

We have

$$M_{n+l}^a(x) = M_l^a(x + n\omega) M_n^a(x),$$

hence

$$-\log \left\| M_l^a(x + n\omega)^{-1} \right\| \leq \log \|M_{n+l}^a(x)\| - \log \|M_n^a(x)\| \leq \log \|M_l^a(x + n\omega)\|$$

for almost every x . Now (3.5) follows by (3.1) and (3.2).

From the fact that

$$M_n^a(x + l\omega) M_l^a(x) = M_l^a(x + n\omega) M_n^a(x)$$

we conclude that

$$\begin{aligned} -\log \left\| M_l^a(x + n\omega)^{-1} \right\| - \log \|M_l^a(x)\| &\leq \log \|M_n^a(x + l\omega)\| - \log \|M_n^a(x)\| \\ &\leq \log \|M_l^a(x + n\omega)\| + \log \left\| M_l^a(x)^{-1} \right\| \end{aligned}$$

for almost every x . Now (3.6) also follows by (3.1) and (3.2).

Let $\lambda \geq \lambda_0$. Then for $k = 1, \dots, l-1$ we have $\lambda l/k > \lambda_0$, so by (3.6) we get

$$|\log \|M_n^a(x + k\omega)\| - \log \|M_n^a(x)\|| \leq \left(\frac{\lambda l}{k} \right) k = \lambda l$$

up to a set of measure less than $\exp(-c\lambda l)$. Summing over $k = 0, \dots, l-1$ and dividing by l we get that (3.7) holds up to a set of measure less than $l \exp(-c\lambda l)$. Finally, note that $l \exp(-c\lambda l) < \exp(-c\lambda l/2)$, $l \geq 1$ if λ is large enough. This concludes the proof. \square

Next we provide bounds on the Riesz representation of $\log \|M_n^a(\cdot)\|$, which are needed to ensure that the constants we will get from Theorem 2.1 don't depend on n .

Lemma 3.4. *Let*

$$\frac{1}{n} \log \|M_n^a(z)\| = \int_{\mathcal{A}_{\rho'_0}} \log |z - \zeta| d\mu_n(\zeta) + h_n(z)$$

be the Riesz representation on $\mathcal{A}_{\rho'_0}$. There exists a constant $C_0 = C_0(\|a\|_\infty, \|b\|_, |E|, \rho_0, \rho'_0, \rho''_0)$ such that*

$$\mu_n(\mathcal{A}_{\rho_0}) + \|h_n\|_{L^\infty(\mathcal{A}_{\rho_0})} \leq C_0.$$

Proof. Let $u_n(z) = \log \|M_n^a(z)\|/n$ and $T_n = \sup_{\mathcal{A}_{\rho'_0}} u_n$. From [GS08, Lemma 2.2] we have that

$$\begin{aligned} \mu_n(\mathcal{A}_{\rho_0}) &\leq \mu_n(\mathcal{A}_{\rho'_0}) \leq C(\rho'_0, \rho''_0) \left(T_n - \sup_{\mathcal{A}_{\rho'_0}} u_n \right) \\ &\leq C \left(T_n - \sup_{\mathbb{T}} u_n \right) \leq C(T_n - L_n^a) \end{aligned}$$

and

$$\begin{aligned} \|h_n\|_{L^\infty(\mathcal{A}_{\rho_0})} &\leq \left\| h_n - \sup_{\mathcal{A}_{\rho'_0}} u_n \right\|_{L^\infty(\mathcal{A}_{\rho_0})} + \sup_{\mathcal{A}_{\rho'_0}} u_n \\ &\leq C(\rho_0, \rho'_0, \rho''_0) \left(T_n - \sup_{\mathcal{A}_{\rho'_0}} u_n \right) + T_n \leq C(T_n - L_n^a) + T_n. \end{aligned}$$

The conclusion now follows from the fact that there exists a constant $C = C(\|a\|_\infty, \|b\|_\infty, |E|, \rho''_0)$ such that $T_n \leq C$, and from (3.8). \square

Now we can prove the first deviations estimate.

Lemma 3.5. *Let $\delta_0 > 0$. For any $\delta \in (0, \delta_0)$ and any integer $n > 1$ we have*

$$\text{mes} \{x \in \mathbb{T} : |\log \|M_n^a(x)\| - nL_n^a| > n\delta\} < \exp(-c_0 n \delta^2 + C_0 (\log n)^p)$$

where $c_0 = c_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \delta_0)$ and $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \omega, p, \delta_0)$.

Proof. We have

$$\begin{aligned} \text{mes} \{x \in \mathbb{T} : |\log \|M_n^a(x)\| - nL_n^a| > n\delta\} &\leq \\ \text{mes} \left\{ x \in \mathbb{T} : \left| \frac{1}{n} \log \|M_n^a(x)\| - \frac{1}{l} \sum_{k=0}^{l-1} \frac{1}{n} \log \|M_n^a(x + k\omega)\| \right| > \frac{\delta}{2} \right\} \\ + \text{mes} \left\{ x \in \mathbb{T} : \left| \frac{1}{l} \sum_{k=0}^{l-1} \frac{1}{n} \log \|M_n^a(x + k\omega)\| - L_n^a \right| > \frac{\delta}{2} \right\}. \quad (3.9) \end{aligned}$$

The conclusion will follow by estimating the two quantities on the right-hand side of the above inequality.

From (3.7) we get

$$\left| \log \|M_n^a(x)\| - \frac{1}{l} \sum_{k=0}^{l-1} \log \|M_n^a(x + k\omega)\| \right| \leq C_1 l$$

up to a set not exceeding $\exp(-cl)$ in measure. Let $l = \lceil \delta n / 2C_1 \rceil + 1$. We have

$$\frac{\delta}{2} < \frac{C_1 l}{n}$$

so we get

$$\left| \frac{1}{n} \log \|M_n^a(x)\| - \frac{1}{l} \sum_{k=0}^{l-1} \frac{1}{n} \log \|M_n^a(x + k\omega)\| \right| \leq \frac{\delta}{2}$$

for all x except for a set of measure less than $\exp(-cl)$. Hence

$$\begin{aligned} \text{mes} \left\{ x \in \mathbb{T} : \left| \frac{1}{n} \log \|M_n^a(x)\| - \frac{1}{l} \sum_{k=0}^{l-1} \frac{1}{n} \log \|M_n^a(x + k\omega)\| \right| > \frac{\delta}{2} \right\} \\ < \exp(-cl) < \exp(-c_1 \delta n), \end{aligned}$$

where $c_1 = c/(2C_1)$.

From Theorem 2.1 we have

$$\begin{aligned} \text{mes} \left\{ x \in \mathbb{T} : \left| \frac{1}{l} \sum_{k=0}^{l-1} \frac{1}{n} \log \|M_n^a(x + k\omega)\| - L_n^a \right| > \frac{\delta}{2} \right\} \\ < \exp \left(-c \frac{\delta}{2} l + C (\log l)^p \right) < \exp \left(-c_2 \delta^2 n + C' (\log n)^p \right). \end{aligned}$$

Recall that Lemma 3.4 ensures that c and C don't depend on n .

Now (3.9) becomes

$$\begin{aligned} \text{mes} \{x \in \mathbb{T} : |\log \|M_n^a(x)\| - nL_n^a| > n\delta\} \\ < \exp(-c_1 \delta n) + \exp(-c_2 \delta^2 n + C' (\log n)^p) \\ < 2 \exp(-c \delta^2 n + C' (\log n)^p) < \exp(-c \delta^2 n + C'' (\log n)^p), \end{aligned}$$

where $c = c(c_1, c_2, \delta_0)$. This concludes the proof. \square

The same proof yields that for $\delta \geq \delta_0$ we have

$$\text{mes} \{x \in \mathbb{T} : |\log \|M_n^a(x)\| - nL_n^a| > n\delta\} < \exp(-c_0 n \delta + C_0 (\log n)^p).$$

For δ_0 large enough, this just follows from (3.4). Also note that to get an estimate when $n = 1$ one just needs to apply Theorem 2.1.

The same type of estimate holds for M_n^u and M_n . We state it only for M_n^u since this is all we need.

Corollary 3.6. *Let $\delta_0 > 0$. For any $\delta \in (0, \delta_0)$ and any integer $n > 1$ we have*

$$\text{mes} \{x \in \mathbb{T} : |\log \|M_n^u(x)\| - nL_n^u| > n\delta\} < \exp(-c_0 n \delta^2 + C_0 (\log n)^p)$$

where $c_0 = c_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \delta_0)$ and $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \omega, p, \delta_0)$.

Proof. Using (2.9) we easily get

$$\begin{aligned} \text{mes} \{x \in \mathbb{T} : |\log \|M_n^u(x)\| - nL_n^u| > n\delta\} \\ \leq \text{mes} \left\{ x \in \mathbb{T} : |\log \|M_n^a(x)\| - nL_n^a| > \frac{n\delta}{2} \right\} \\ + \text{mes} \left\{ x \in \mathbb{T} : |\tilde{S}_n(x) - n\tilde{D}| > \frac{n\delta}{2} \right\} + \text{mes} \left\{ x \in \mathbb{T} : |S_n(x + \omega) - nD| > \frac{n\delta}{2} \right\}. \end{aligned}$$

The conclusion now follows from Lemma 3.5 and Theorem 2.1. \square

The next step is to make use of the Avalanche Principle to improve the previous estimate. The following lemma is the most general application of the Avalanche Principle that suits our purposes.

Lemma 3.7. *Let $n > 1$ be an integer and $n = \sum_{j=1}^m l_j$ where l_j are positive integers such that $l \leq l_j \leq 3l$, with $l = l(n)$ a real number. Let $A_j(x) = A_j(x, n)$ be 2×2 matrices for $x \in \mathbb{T}$, and let $L_k, k \geq 1$ be a sequence of real numbers. If*

$$l > \frac{2}{\gamma} \log n,$$

$$L_{l_j} - L_{l_j+l_{j+1}} \leq \frac{\gamma}{100}, L_{l_{j+1}} - L_{l_j+l_{j+1}} \leq \frac{\gamma}{100}, j = 1, \dots, m-1$$

$$\max_{1 \leq j \leq m} |\det A_j(x)| \leq 1, \text{ a.e. } x \in \mathbb{T},$$

$$\text{mes} \left\{ x \in \mathbb{T} : \left| \frac{1}{l_j} \log \|A_j(x)\| - L_{l_j} \right| > \frac{\gamma}{100} \right\} \leq \exp(-c_0 l_j^\sigma), j = 1, \dots, m,$$

and

$$\begin{aligned} \text{mes} \left\{ x \in \mathbb{T} : \left| \frac{1}{l_j + l_{j+1}} \log \|A_{j+1}(x) A_j(x)\| - L_{l_j+l_{j+1}} \right| > \frac{\gamma}{100} \right\} \\ \leq \exp(-c_0 (l_j + l_{j+1})^\sigma), j = 1, \dots, m-1, \end{aligned}$$

then there exists an absolute constant C_0 such that

$$\begin{aligned} \left| \log \|A_m(x) \dots A_1(x)\| + \sum_{j=2}^{m-1} \log \|A_j(x)\| - \sum_{j=1}^{m-1} \log \|A_{j+1}(x) A_j(x)\| \right| \\ < C_0 m \exp\left(-\frac{\gamma}{2} l\right) < C_0 \frac{1}{l} \end{aligned}$$

up to a set of measure less than $3n \exp(-c_0 l^\sigma)$.

Proof. Let $\mu = \exp(l\gamma/2)$. We have

$$\min_{1 \leq j \leq m} \|A_j(x)\| \geq \min_{1 \leq j \leq m} \exp\left(l_j L_{l_j} - \frac{\gamma}{100}\right) > \exp(l\gamma/2) = \mu > n$$

and

$$\begin{aligned} \max_{1 \leq j < m-1} [\log \|A_{j+1}(x)\| + \log \|A_j(x)\| - \log \|A_{j+1}(x) A_j(x)\|] \\ \leq l_{j+1} \left(L_{l_{j+1}} + \frac{\gamma}{100} \right) + l_j \left(L_{l_j} + \frac{\gamma}{100} \right) - (l_j + l_{j+1}) \left(L_{l_j+l_{j+1}} - \frac{\gamma}{100} \right) \\ = l_{j+1} \left(L_{l_{j+1}} - L_{l_j+l_{j+1}} + \frac{2\gamma}{100} \right) + l_j \left(L_{l_j} - L_{l_j+l_{j+1}} + \frac{2\gamma}{100} \right) \\ < 6l \frac{3\gamma}{100} < \frac{\gamma l}{4} = \frac{1}{2} \log \mu \end{aligned}$$

up to a set of measure $3m \exp(-c_0 l^\sigma) < 3n \exp(-c_0 l^\sigma)$. The conclusion follows from the Avalanche Principle and the fact that $m/\mu < 1/l$. \square

As mentioned before, it is important for us that the constants in the deviations estimate can be chosen uniformly for E in a compact set. For this we need to provide a bound for $L_n^u - L^u$ that holds for all E in a compact set. First we state a simple estimate that we will use to deal with the integrals over the exceptional sets for our functions.

Lemma 3.8. *Let f be a measurable function defined on \mathbb{T} such that for any $\delta \geq \delta_0$ we have that $|f(x)| \leq \delta$ up to a set of measure less than $\exp(-c_0\delta)$. Then $\|f\|_{L^2(\mathbb{T})} \leq C_0$, where $C_0 = C_0(c_0, \delta_0)$.*

Lemma 3.9. *For any integer $n > 1$ we have*

$$0 \leq L_n - L = L_n^u - L^u = L_n^a - L^a < C_0 \frac{(\log n)^2}{n}$$

where $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \gamma)$.

Proof. It is sufficient to get the estimate for large n . We will tacitly assume that n is large enough for our estimates to hold. We should keep in mind that the choice of large n should be uniform for E in a bounded set.

It is easy to see that the conclusion follows if we have

$$|L_{2n}^a - L_n^a| \leq C \frac{(\log n)^2}{n}. \quad (3.10)$$

Since we have

$$|L_{2n}^a - L_n^a| = \left| \int_{\mathbb{T}} \frac{\log \|M_{2n}^a(x)\| - \log \|M_n^a(x + n\omega)\| - \log \|M_n^a(x)\|}{2n} dx \right|,$$

it will be sufficient to prove that

$$|\log \|M_{2n}^a(x)\| - \log \|M_n^a(x + n\omega)\| - \log \|M_n^a(x)\|| \leq C_1 (\log n)^2 \quad (3.11)$$

up to a set not exceeding $C_2 n^{-1}$ in measure. Indeed, from (3.1) it follows that for $\delta \geq \delta_0$ we have

$$\left| \frac{\log \|M_{2n}^a(x)\| - \log \|M_n^a(x + n\omega)\| - \log \|M_n^a(x)\|}{2n} \right| \leq \delta$$

up to a set not exceeding $\exp(-c_1\delta 2n) + 2\exp(-c_1\delta n) < \exp(-c\delta n)$ in measure, and by using (3.11) and Lemma 3.8 we get

$$\begin{aligned} |L_{2n}^a - L_n^a| &\leq \int_{\mathbb{T}} \left| \frac{\log \|M_{2n}^a(x)\| - \log \|M_n^a(x + n\omega)\| - \log \|M_n^a(x)\|}{2n} \right| dx \\ &\leq C_1 (\log n)^2 + C_3 \sqrt{C_2 n^{-1}} \leq C (\log n)^2. \end{aligned}$$

Now we check that the sufficient condition (3.11) holds. Let $l = [C_l \log n]$ and $m = [n/l]$. If C_l is sufficiently large we have that $l > 2 \log n / \gamma$ and $3n \exp(-cl) < n^{-1}$. We want to choose C_l so that $L_l^u - L_{2l}^u \leq \gamma/100$ and $C_l \leq C$ (note that

without the bound, such C_l obviously exists). Suppose that $L_{2^j l}^u - L_{2^{j+1} l}^u > \frac{\gamma}{100}$ for $j \geq 0$. Then using (3.8) we get

$$C - \frac{\tilde{D} + D}{2} \geq L_l^u - L_{2^{j+1} l}^u > \frac{j\gamma}{100}.$$

This shows that by eventually replacing l with $2^j l$ with some

$$j < 100(2C - \tilde{D} - D)/\gamma$$

we will have $L_l^u - L_{2l}^u \leq \gamma/100$, and the corresponding C_l will be bounded. Using Corollary 3.6 and Lemma 3.7 we get

$$\left| \log \|M_{ml}^a(x)\| + \sum_{j=1}^{m-2} \log \|M_l^a(x + jl\omega)\| - \sum_{j=0}^{m-2} \log \|M_{2l}^a(x + jl\omega)\| \right| < C \quad (3.12)$$

up to a set not exceeding n^{-1} in measure, and analogous estimates for $\log \|M^a(x + ml\omega)\|$ and $\log \|M_{2ml}^a(x)\|$. Recall that we apply the Avalanche Principle to M_n^u but the conclusion also holds for M_n^a . Note that we need to have $m \geq 2$. This clearly holds for large enough n depending on C_l . This can be done uniformly for E in a bounded set because of our bound on C_l . Putting these estimates together we get

$$\begin{aligned} & \left| \log \|M_{2ml}^a(x)\| - \log \|M_{ml}^a(x + ml\omega)\| - \log \|M_{ml}^a(x)\| \right. \\ & \quad \left. + \log \|M_l^a(x + (m-1)l\omega)\| + \log \|M_l^a(x + ml\omega)\| \right. \\ & \quad \left. - \log \|M_{2l}^a(x + (m-1)l\omega)\| \right| < C \quad (3.13) \end{aligned}$$

up to a set not exceeding Cn^{-1} in measure. By (3.1) we have that $|\log \|M_l^a(x)\|| \leq C \log n$ up to a set not exceeding n^{-1} in measure. From this, similar estimates, and (3.13) we get

$$|\log \|M_{2ml}^a(x)\| - \log \|M_{ml}^a(x + ml\omega)\| - \log \|M_{ml}^a(x)\|| < C \log n \quad (3.14)$$

up to a set not exceeding Cn^{-1} in measure.

From (3.5) we get that for sufficiently large δ we have

$$|\log \|M_n^a(x)\| - \log \|M_{ml}^a(x)\|| \leq \delta(n - ml)$$

up to a set not exceeding $\exp(-c\delta(n - ml))$ in measure. We can choose $\delta > (\log n)/c$ to conclude that

$$|\log \|M_n^a(x)\| - \log \|M_{ml}^a(x)\|| \leq C(\log n)^2$$

up to a set not exceeding n^{-1} in measure. From this, similar estimates (using (3.5) and (3.6)), and (3.14) we can conclude that

$$|\log \|M_{2n}^a(x)\| - \log \|M_n^a(x + n\omega)\| - \log \|M_n^a(x)\|| < C(\log n)^2$$

up to a set not exceeding Cn^{-1} in measure. Thus we proved (3.11) and this concludes the proof. \square

The bound from the previous lemma can be improved, as in [GS01, Theorem 5.1], to $L_n - L \leq C_0/n$. However, we won't need this better bound in this paper.

Now we are able to prove the improved version of the deviations estimate (cf. [GS01, Theorem 7.1]).

Theorem 3.10. *For any $\delta > 0$ and any integer $n > 1$ we have*

$$\text{mes} \{x \in \mathbb{T} : |\log \|M_n^a(x)\| - nL_n^a| > \delta n\} < \exp(-c_0\delta n + C_0(\log n)^p)$$

where $c_0 = c_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \gamma)$ and $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \gamma, p)$. The same estimate, with possibly different constants, holds with L^a instead of L_n^a .

Proof. First note that due to (3.4) we just need to check the estimate for $\delta < \delta_0$. Furthermore, note that the estimate is trivial if $-c_0\delta n + C_0(\log n)^p > 0$. Hence we just need to check the estimate for δ satisfying

$$C \frac{(\log n)^p}{n} \leq \delta < \delta_0, \quad (3.15)$$

where $C = C_0/c_0$ can be made as large as we need by choosing C_0 large. Furthermore by choosing C_0 large enough we can make sure that the deviations estimate holds trivially for small n . Hence it is enough to check the estimate for n large enough.

Let $l = [\delta n] + 1$, $m = [n/l]$ and $l' = n - (m-1)l$. An application of the Avalanche Principle (using Corollary 3.6, (3.15), and Lemma 3.7) yields

$$\begin{aligned} \log \|M_n^a(x)\| + \sum_{j=1}^{m-2} \log \|M_l^a(x + jl\omega)\| - \log \|M_{l'+l}^a(x + (m-2)l\omega)\| \\ - \sum_{j=0}^{m-3} \log \|M_l^a(x + (j+1)l\omega) M_l^a(x + jl\omega)\| = O\left(\frac{1}{l}\right) \end{aligned}$$

up to a set of measure less than $3n \exp(-cl) < \exp(-c\delta n/2)$. From (3.1) we can conclude that

$$\begin{aligned} |\log \|M_{l'}^a(x + (m-1)l\omega) M_l^a(x + (m-2)l\omega)\|| \\ = |\log \|M_{l'+l}^a(x + (m-2)l\omega)\|| \leq Cl \end{aligned}$$

up to a set of measure less than $\exp(-cl) \leq \exp(-c\delta n)$. Hence

$$\log \|M_n^a(x)\| + \sum_{j=1}^{m-2} \log \|M_l^a(x + jl\omega)\| - \sum_{j=0}^{m-3} \log \|M_{2l}^a(x + jl\omega)\| = O(l)$$

up to a set of measure less than $\exp(-c\delta n)$. Summing the above estimate with

$x + k\omega$ instead of x yields

$$\begin{aligned} \frac{1}{l} \sum_{k=0}^{l-1} \log \|M_n^a(x + k\omega)\| + \sum_{j=l}^{(m-1)l-1} \frac{1}{l} \log \|M_l^a(x + j\omega)\| \\ - \sum_{j=0}^{(m-2)l-1} \frac{1}{l} \log \|M_{2l}^a(x + j\omega)\| = O(l) \end{aligned}$$

up to a set of measure less than $l \exp(-c\delta n) < \exp(-c\delta n/2)$. Using (3.7) we can conclude that

$$\log \|M_n^a(x)\| + \sum_{j=l}^{(m-1)l-1} \frac{1}{l} \log \|M_l^a(x + j\omega)\| - \sum_{j=0}^{(m-2)l-1} \frac{1}{l} \log \|M_{2l}^a(x + j\omega)\| = O(l)$$

up to a set of measure less than $\exp(-c_1\delta n) + \exp(-c_2l) < \exp(-c\delta n)$. From this, Theorem 2.1, and (3.8) it follows that

$$\log \|M_n^a(x)\| + (m-2)l(L_l^a - 2L_{2l}^a) = O(\delta n)$$

up to a set of measure less than

$$2 \exp(-c_1\delta n + C(\log n)^p) + \exp(-c_2\delta n) < \exp(-c\delta n + C(\log n)^p).$$

Integrating over \mathbb{T} and using Lemma 3.8 yields

$$|nL_n^a + (m-2)l(L_l^a - 2L_{2l}^a)| < C_1\delta n + C_2n \exp((-c\delta n + C(\log n)^p)/2) < C\delta n.$$

Note that for the last inequality to hold we need to choose C large enough in (3.15). Now we have that

$$|\log \|M_n^a(x)\| - nL_n^a| < C\delta n$$

up to a set of measure less than $\exp(-c\delta n + C(\log n)^p)$. The fact that L_n^a can be replaced by L^a follows from Lemma 3.9 and (3.15). \square

Corollary 3.11. *For any $\delta > 0$ and any integer $n > 1$ we have*

$$\text{mes}\{x \in \mathbb{T} : |\log \|M_n^u(x)\| - nL_n^u| > \delta n\} < \exp(-c_0\delta n + C_0(\log n)^p)$$

where $c_0 = c_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \gamma)$ and $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \gamma, p)$. The same estimate, with possibly different constants, holds with L^u instead of L_n^u .

Proof. The proof is the same as for Corollary 3.6. \square

Next we establish some estimates that will be needed in the next section. First we prove a uniform upper bound for $\log \|M_n^a\|$. We will need the following general result about averages of subharmonic functions.

Lemma 3.12. ([GS08, Lemma 4.1]) Let u be a subharmonic function and let

$$u(z) = \int_{\mathbb{C}} \log |z - \zeta| d\mu(\zeta) + h(z)$$

be its Riesz representation on a neighborhood of \mathcal{A}_ρ . If $\mu(\mathcal{A}_\rho) + \|h\|_{L^\infty(\mathcal{A}_\rho)} \leq M$ then for any $r_1, r_2 \in (1 - \rho, 1 + \rho)$ we have

$$|\langle u(r_1(\cdot)) \rangle - \langle u(r_2(\cdot)) \rangle| \leq C_0 |r_1 - r_2|,$$

where $C_0 = C_0(M, \rho)$.

The following corollary is an immediate consequence of the previous lemma and Lemma 3.4.

Corollary 3.13. There exists a constant $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \rho_0, \rho'_0, \rho''_0)$ such that

$$|L_n^u(r_1) - L_n^u(r_2)| = |L_n^a(r_1) - L_n^a(r_2)| \leq C_0 |r_1 - r_2|$$

for any $r_1, r_2 \in (1 - \rho_0, 1 + \rho_0)$ and any positive integer n .

Proposition 3.14. For any integer $n > 1$ we have that

$$\sup_{x \in \mathbb{T}} \log \|M_n^a(x)\| \leq nL_n^a + C_0 (\log n)^p$$

where $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E|, \omega, \gamma, p)$.

Proof. It is sufficient to establish the estimate for large n . From the large deviations estimate, with $n\delta = C(\log n)^p$ where C is sufficiently large, we have

$$\log \|M_n^a(rx)\| - nL_n^a(r) \leq C(\log n)^p$$

except for a set $\mathcal{B}(r)$ of measure less than $\exp(-c_1 C(\log n)^p + C'(\log n)^p) < \exp(-c(\log n)^p)$. Here r is in a neighborhood of 1 such that $L^u(r) \geq \gamma/2$. Such a neighborhood exists because of Corollary 3.13. By the subharmonicity of $\log \|M_n^a(z)\|$ we have

$$\begin{aligned} \log \|M_n^a(x)\| - nL_n^a &\leq \frac{1}{\pi n^{-2}} \int_{D(x, n^{-1})} (\log \|M_n^a(z)\| - nL_n^a) dA(z) \\ &\leq \frac{1}{\pi n^{-2}} \int_{1-n^{-1}}^{1+n^{-1}} \int_{x-2n^{-1}}^{x+2n^{-1}} |\log \|M_n^a(ry)\| - L_n^a| r dy dr. \end{aligned} \quad (3.16)$$

For $r \in (1 - n^{-1}, 1 + n^{-1})$ we have

$$\begin{aligned} &\int_{x-2n^{-1}}^{x+2n^{-1}} |\log \|M_n^a(ry)\| - L_n^a| dy \\ &\leq \int_{x-2n^{-1}}^{x+2n^{-1}} |\log \|M_n^a(ry)\| - L_n^a(r)| dy + |L_n^a - L_n^a(r)| \\ &\leq C_1 (\log n)^p n^{-1} + C_2 n \exp(-c(\log n)^p/2) + C_3 n^{-1} < C(\log n)^p n^{-1}. \end{aligned}$$

As usual, we used Lemma 3.8 to deal with the exceptional set. Plugging this estimate in (3.16) yields the desired conclusion. \square

As was mentioned in the introduction, from this point forward we will make use of the fact that $\tilde{b} = \bar{b}$ on \mathbb{T} . In particular we will tacitly use that $D = \tilde{D}$, $S = \tilde{S}$, $L_n = L_n^u$, $L = L^u$, and $|\tilde{b}| = |\bar{b}| = |b|$.

Next we want to estimate $L_n(E) - L_n(E_0)$ in a neighborhood of E_0 .

Lemma 3.15. *There exist constants $C_0 = C_0(\|a\|_\infty, \|b\|_*, \max\{|E_1|, |E_2|\})$ and $c_0 = c_0(\|b\|_*, \omega)$ such that*

$$\begin{aligned} & |\log \|M_l^u(x, E_1)\| - \log \|M_l^u(x, E_2)\|| \\ &= |\log \|M_l^a(x, E_1)\| - \log \|M_l^a(x, E_2)\|| \leq \exp(C_0 l) |E_1 - E_2| \end{aligned}$$

holds for any positive integer l and any x up to a set (independent of E_1 and E_2) of measure less than $\exp(-c_0 l)$.

Proof. The identity follows from (2.9). By the Mean Value Theorem we have

$$\begin{aligned} & |\log \|M_l^a(x, E_1)\| - \log \|M_l^a(x, E_2)\|| \\ &\leq \frac{1}{\min\{\|M_l^a(x, E_1)\|, \|M_l^a(x, E_2)\|\}} \|\|M_l^a(x, E_1)\| - \|M_l^a(x, E_2)\|\| \\ &\leq \frac{1}{\min\{\|M_l^a(x, E_1)\|, \|M_l^a(x, E_2)\|\}} \sup_{E \in [E_1, E_2]} \left\| \frac{\partial}{\partial E} M_l^a(x, E) \right\| |E_1 - E_2|. \end{aligned}$$

There exists a constant $C = C(\|a\|_\infty, \|b\|_\infty, \max\{|E_1|, |E_2|\})$ such that

$$\sup_{E \in [E_1, E_2]} \left\| \frac{\partial}{\partial E} M_l^a(x, E) \right\| \leq \exp(Cl).$$

The conclusion now follows by using (3.1). \square

Lemma 3.16. *Fix $E_0 \in \mathbb{C}$ such that $L(E_0) \geq \gamma$. There exist constants $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E_0|, \omega, \gamma)$, $C_1 = C_1(\|a\|_\infty, \|b\|_*, |E_0|, \omega, \gamma)$, and $n_0 = n_0(\|a\|_\infty, \|b\|_*, |E_0|, \omega, \gamma)$ such that we have*

$$|\log \|M_n^a(x, E)\| - \log \|M_n^a(x, E_0)\|| \leq n^{-C_0}$$

for $n \geq n_0$, $|E - E_0| < n^{-C_1}$, and all x up to a set $\mathcal{B} = \mathcal{B}(n, E_0)$ of measure less than n^{-1} .

Proof. Let $l = \lceil C_2 \log n \rceil$, $m = \lfloor n/l \rfloor$, and $l' = n - (m-2)l$. In what follows we should keep in mind that some of the estimates hold by choosing C_2 large enough. To be able to apply the Avalanche Principle we will need that $m \geq 2$, hence we also need that n is large enough. Applying the Avalanche Principle (see Lemma 3.7) we get

$$\begin{aligned} & \log \|M_n^a(x, E_0)\| + \sum_{j=1}^{m-2} \log \|M_l^a(x + jl\omega, E_0)\| - \log \|M_{l+l'}^a(x + (m-2)l\omega, E_0)\| \\ &= \sum_{j=0}^{m-3} \log \|M_{2l}^a(x + jl\omega, E_0)\| = O\left(\frac{n}{l} \exp\left(-\frac{\gamma}{2}l\right)\right) = O\left(\frac{1}{n^{cC_2}}\right) \quad (3.17) \end{aligned}$$

up to a set of measure $3n \exp(-c_1 l) < n^{-cC_2}$. We claim that the Avalanche Principle can be applied, with the same μ , for the same factorization of $M_n^a(x, E)$. Note that we cannot apply the deviations estimate since we don't know whether $L(E) > 0$. For example, Lemma 3.15 and Corollary 3.6 imply that

$$\begin{aligned} \log \|M_l^u(x, E)\| &\geq \log \|M_l^u(x, E_0)\| - \exp(Cl - C_1 \log n) \\ &\geq \left(\gamma - \frac{\gamma}{100}\right) l - \exp(Cl - C_1 \log n) > \frac{\gamma}{2} l \end{aligned}$$

up to a set of measure $\exp(-c_1 l) + \exp(-c_2 l) < \exp(-cl)$. Note that the exceptional set from the deviation estimate is already included in the exceptional set for (3.17) and recall that the exceptional set from Lemma 3.15 doesn't depend on E . Also note that C_1 needs to satisfy $C_1 \geq CC_2$. The other estimates needed for the Avalanche Principle are obtained similarly, provided C_1 is large enough. Hence, (3.17) holds with E instead of E_0 . The conclusion follows by subtracting (3.17) for E and E_0 and using Lemma 3.15 (again, C_1 needs to be chosen to be large enough). \square

Corollary 3.17. *Fix $E_0 \in \mathbb{C}$ such that $L(E_0) \geq \gamma$. There exist constants $C_0 = C_0(\|a\|_\infty, \|b\|_*, |E_0|, \omega, \gamma)$, $C_1 = C_1(\|a\|_\infty, \|b\|_*, |E_0|, \omega, \gamma)$, and $n_0 = n_0(\|a\|_\infty, \|b\|_*, |E_0|, \omega, \gamma)$ such that we have*

$$|n(L_n(E) - L_n(E_0))| = |n(L_n^a(E) - L_n^a(E_0))| \leq n^{-C_0}$$

for $n \geq n_0$ and $|E - E_0| < n^{-C_1}$.

Proof. Integrate the estimate of the previous lemma. To deal with the exceptional set we used Lemma 3.8 and the fact that as a consequence of (3.1) we have

$$|\log \|M_n^a(x, E)\| - \log \|M_n^a(x, E_0)\|| \leq \lambda n$$

up to a set of size $\exp(-c\lambda n)$ for any $\lambda \geq \lambda_0$. \square

4 Estimates for the Entries of the Fundamental Matrix

We will need the following particular case of a lemma from [GS08].

Lemma 4.1. ([GS08, Lemma 2.4]) *Let u be a subharmonic function defined on \mathcal{A}_ρ such that $\sup_{\mathcal{A}_\rho} u \leq M$. There exist constants $C_1 = C_1(\rho)$ and C_2 such that, if for some $0 < \delta < 1$ and some L we have*

$$\text{mes}\{x \in \mathbb{T} : u(x) < -L\} > \delta,$$

then

$$\sup_{\mathbb{T}} u \leq C_1 M - \frac{L}{C_1 \log(C_2/\delta)}.$$

Let $I_{a,E} = \int_{\mathbb{T}} \log |a(x) - E| dx$. Note that $|I_{a,E}| < \infty$ if and only if $a \not\equiv E$. If $a \equiv E$ then it is straightforward to see that $L = 0$. Hence if $L(E) > 0$ then $|I_{a,E}| < \infty$. Furthermore, if $L(E) > 0$ on some set, it can be seen that $I_{a,E}$ is continuous in E on that set.

Lemma 4.2. *There exists $l_0 = l_0(\|a\|_{\infty}, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ such that*

$$\text{mes} \{x \in \mathbb{T} : |f_l(x)| \leq \exp(-l^3)\} \leq \exp(-l)$$

for all $l \geq l_0$.

Proof. We argue by contradiction. Assume

$$\text{mes} \{x \in \mathbb{T} : |f_l(x)| \leq \exp(-l^3)\} > \exp(-l)$$

for arbitrarily large l . We will be tacitly using the fact that l can be arbitrarily large. We have that

$$|f_l^a(x)| = |f_l(x)| \prod_{j=1}^l |b(x + j\omega)| \leq \exp(-l^3) C^{l-1} \leq \exp(-l^3/2)$$

on a set of measure greater than $\exp(-l)$. Hence

$$\text{mes} \{x \in \mathbb{T} : |f_l^a(x)| \leq \exp(-l^3/2)\} > \exp(-l).$$

At the same time we have that

$$\log |f_l^a(x)| \leq \log \|M_l^a(x)\| \leq Cl$$

for all x , so by applying Lemma 4.1 we get that

$$|f_l^a(x)| \leq \exp \left(C_1 l - \frac{l^3}{C_2 \log(C_3 \exp(l))} \right) \leq \exp(-Cl^2)$$

for all x and consequently

$$|f_l(x)| \leq \exp((l-1)(1-D) - C_1 l^2) \leq \exp(-Cl^2) \quad (4.1)$$

for all x except for a set of measure less than $\exp(-c_1(l-1) + r_{l-1}) < \exp(-cl)$.

From Corollary 3.11 we have that

$$\begin{aligned} \exp(l\gamma) \leq \|M_l(x)\|^2 &\leq 2 \left(|f_l(x)|^2 + |f_{l-1}(x)|^2 \right. \\ &\quad \left. + \left| \frac{b(x)}{b(x+\omega)} f_{l-1}(x+\omega) \right|^2 + \left| \frac{b(x)}{b(x+\omega)} f_{l-2}(x+\omega) \right|^2 \right) \end{aligned} \quad (4.2)$$

for all x except for a set of measure less than $\exp(-c_1\gamma l/2 + r_l) < \exp(-cl)$.

Suppose that

$$\left| \frac{b(x)}{b(x+\omega)} f_{l-1}(x+\omega) \right|^2 \geq \frac{1}{4} \exp(l\gamma) \quad (4.3)$$

for all x except for a set of measure less than $1/3$ (any constant in $(0, 1/2)$ would work). Since

$$\begin{aligned} \frac{\overline{b(x)}}{b(x+l\omega)} = \det M_l(x) &= -\frac{\overline{b(x)}}{b(x+\omega)} f_l(x) f_{l-2}(x+\omega) \\ &\quad + \frac{\overline{b(x)}}{b(x+\omega)} f_{l-1}(x) f_{l-1}(x+\omega) \end{aligned}$$

it follows that

$$\begin{aligned} |f_{l-1}(x)| &= \left| \frac{b(x)}{b(x+\omega)} f_{l-1}(x+\omega) \right|^{-1} \left| \frac{b(x)}{b(x+l\omega)} + \frac{b(x)}{b(x+\omega)} f_l(x) f_{l-2}(x+\omega) \right| \\ &\leq 2 \exp(-l\gamma/2) (C_1 \exp(\delta - D) + \exp(-C_2 l^2 + C_3 l)) \end{aligned}$$

for all x except for a set of measure less than $1/3 + \exp(-c_1\delta + r_1) + \exp(-c_1l) + \exp(-c_2l + r'_l)$. Note that in the above estimate we used

$$\log \left| \frac{b(x)}{b(x+\omega)} f_{l-2}(x+\omega) \right| \leq \log \|M_l(x)\|$$

and the large deviations estimate for M_l . Choosing $\delta = l\gamma/2$ we get

$$|f_{l-1}(x)| \leq C$$

for all x except for a set of measure less than $1/3 + \exp(-cl)$. This contradicts (4.3) because

$$|f_{l-1}(x+\omega)| \leq C$$

and (4.3) would hold at the same time on a set of measure greater than $1/3 - \exp(-cl)$. Hence we must have

$$\left| \frac{b(x)}{b(x+\omega)} f_{l-1}(x+\omega) \right|^2 < \frac{1}{4} \exp(l\gamma) \quad (4.4)$$

on a set of measure greater than $1/3$. At the same time

$$\begin{aligned} \exp((l+1)\gamma) &\leq \|M_{l+1}(x)\|^2 \\ &\leq 2 \left(|f_{l+1}(x)|^2 + |f_l(x)|^2 + \left| \frac{b(x)}{b(x+\omega)} f_l(x+\omega) \right|^2 + \left| \frac{b(x)}{b(x+\omega)} f_{l-1}(x+\omega) \right|^2 \right) \end{aligned}$$

for all x except for a set of measure less than

$$\exp(-c_1\gamma(l+1)/2 + r_{l+1}) < \exp(-cl).$$

This, (4.1), and (4.4) imply that we must have

$$\begin{aligned} |f_{l+1}(x)|^2 &\geq \frac{1}{2} \exp((l+1)\gamma) - \exp(-C_1 l^2) \\ &\quad - C_2 \exp(l - D - C_1 l^2) - \frac{1}{4} \exp(l\gamma) > \frac{1}{4} \exp(l\gamma) \end{aligned}$$

on a set of measure greater than

$$\frac{1}{3} - \exp(-c_1 l) - 2 \exp(-c_2 l) - \exp(-c_3 l + r_1) > \frac{1}{3} - \exp(-cl).$$

From

$$\begin{aligned} \frac{\overline{b(x)}}{b(x + (l+1)\omega)} = \det M_{l+1}(x) &= -\frac{\overline{b(x)}}{b(x + \omega)} f_{l+1}(x) f_{l-1}(x + \omega) \\ &\quad + \frac{\overline{b(x)}}{b(x + \omega)} f_l(x) f_l(x + \omega) \end{aligned}$$

it can be seen that

$$\begin{aligned} \left| \frac{b(x)}{b(x + \omega)} f_{l-1}(x + \omega) \right| &= |f_{l+1}(x)|^{-1} \\ &\cdot \left| \frac{b(x)}{b(x + (l+1)\omega)} - \frac{b(x)}{b(x + \omega)} f_l(x) f_l(x + \omega) \right| \\ &\leq 2 \exp(-l\gamma/2) (C_1 \exp(\delta - D) + C_1 \exp(\delta - D - C_2 l^2)) \end{aligned}$$

on a set of measure greater than $1/3 - \exp(-c_1 l) - 2 \exp(-c_2 \delta + r_1) - 2 \exp(-c_3 l)$.
Choosing $\delta = l\gamma/5$ we get

$$\left| \frac{b(x)}{b(x + \omega)} f_{l-1}(x + \omega) \right| \leq \exp(-l\gamma/4) \quad (4.5)$$

on a set of measure greater than $1/3 - \exp(-cl)$. We will contradict (4.2) by showing that

$$|f_l(x)|^2 + |f_{l-1}(x)|^2 + \left| \frac{b(x)}{b(x + \omega)} f_{l-1}(x + \omega) \right|^2 + \left| \frac{b(x)}{b(x + \omega)} f_{l-2}(x + \omega) \right|^2 \leq C \quad (4.6)$$

on a set of measure greater than $1/3 - \exp(-cl)$. Let G_l be the set on which (4.5) holds.

By writing

$$M_l(x + \omega) = \frac{1}{b(x + (l+1)\omega)} \begin{bmatrix} a(x + l\omega) - E & -\overline{b(x + l\omega)} \\ b(x + (l+1)\omega) & 0 \end{bmatrix} M_{l-1}(x + \omega)$$

we get

$$f_l(x + \omega) = \frac{a(x + l\omega) - E}{b(x + (l+1)\omega)} f_{l-1}(x + \omega) - \frac{\overline{b(x + l\omega)}}{b(x + (l+1)\omega)} f_{l-2}(x + \omega).$$

From this we deduce that

$$\begin{aligned} \left| \frac{b(x)}{b(x + \omega)} f_{l-2}(x + \omega) \right| &= \left| \frac{b(x + (l+1)\omega)}{b(x + l\omega)} \right| \\ &\cdot \left| \frac{a(x + l\omega) - E}{b(x + (l+1)\omega)} \frac{b(x)}{b(x + \omega)} f_{l-1}(x + \omega) - \frac{b(x)}{b(x + \omega)} f_l(x + \omega) \right| \\ &\leq C_1 \exp(\delta - D) (C_2 \exp(\delta - D - \gamma l/4) + C_1 \exp(\delta - D - C_3 l^2)) \end{aligned}$$

on a subset of G_l of measure greater than

$$\frac{1}{3} - 3 \exp(-c_1 \delta + r_1) - \exp(-c_2 l) - \exp(-c_3 l).$$

By choosing $\delta = \gamma l / 17$ we get

$$\left| \frac{b(x)}{b(x+\omega)} f_{l-2}(x+\omega) \right| \leq \exp(-\gamma l / 8)$$

on a subset of G_l of measure greater than $1/3 - \exp(-cl)$.

By writing

$$M_l(x-\omega) = M_{l-1}(x) \frac{1}{b(x)} \begin{bmatrix} a(x-\omega) - E & -\overline{b(x-\omega)} \\ b(x) & 0 \end{bmatrix}$$

we get

$$f_l(x-\omega) = \frac{a(x-\omega) - E}{b(x)} f_{l-1}(x) - \frac{\overline{b(x)}}{b(x+\omega)} f_{l-2}(x+\omega).$$

From this we deduce that

$$\begin{aligned} |f_{l-1}(x)| &= \left| \frac{a(x-\omega) - E}{b(x)} \right|^{-1} \left| f_l(x-\omega) + \frac{\overline{b(x)}}{b(x+\omega)} f_{l-2}(x+\omega) \right| \\ &\leq C_1 \exp(\delta - I_{a,E}) (\exp(-C_1 l^2) + \exp(-\gamma l / 8)) \end{aligned}$$

on a subset of G_l of measure greater than $1/3 - \exp(-c_1 \delta + r_1) - \exp(-c_3 l) - \exp(-c_4 l)$. By choosing $\delta = \gamma l / 17$ we get

$$|f_{l-1}(x)| \leq \exp(-\gamma l / 16)$$

on a subset of G_l of measure greater than $1/3 - \exp(-cl)$. Now it is easy to see that we have (4.6). \square

Lemma 4.3. *Let $\sigma > 0$. There exist constants $l_0 = l_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma, \sigma)$ and $N_0 = N_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma, \sigma)$ such that*

$$\text{mes} \{x \in \mathbb{T} : |f_l(x)| \leq \exp(-N^\sigma)\} \leq \exp(-N^\sigma l^{-2})$$

for any $N \geq N_0$ and for any $l_0 \leq l \leq N^{\sigma/3}$. The same result, but with possibly different l_0 and N_0 , holds for f_l^u .

Proof. We argue by contradiction. Assume

$$\text{mes} \{x \in \mathbb{T} : |f_l(x)| \leq \exp(-N^\sigma)\} > \exp(-N^\sigma l^{-2})$$

for some arbitrarily large l and N . We have that

$$|f_l^a(x)| = |f_l(x)| \prod_{j=1}^l |b(x+j\omega)| \leq \exp(-N^\sigma) C^{l-1} \leq \exp(-N^\sigma / 2)$$

on a set of measure greater than $\exp(-N^\sigma l^{-2})$. Hence

$$\text{mes} \{x \in \mathbb{T} : |f_l^a(x)| \leq \exp(-N^\sigma/2)\} > \exp(-N^\sigma l^{-2}).$$

By applying Lemma 4.1 we get that

$$|f_l^a(x)| \leq \exp\left(C_1 l - \frac{N^\sigma}{2C_1 \log(C_2 \exp(N^\sigma l^{-2}))}\right) \leq \exp(-Cl^2)$$

for all x . Note that the last inequality is equivalent to

$$\frac{C_1}{l} + C \leq \frac{N^\sigma l^{-2}}{2C_1 \log(C_2 \exp(N^\sigma l^{-2}))} = \frac{N^\sigma l^{-2}}{2C_1 \log C_2 + 2C_1 N^\sigma l^{-2}}$$

which clearly holds with $C = 1/(4C_1)$ for large l and N , since $N^\sigma l^{-2} \geq N^{\sigma/3}$. We now have that

$$|f_l(x)| \leq \exp((l-1)(1-D) - C'l^2) \leq \exp(-Cl^2)$$

for all x except for a set of measure less than $\exp(-c_1(l-1) + r_{l-1}) < \exp(-cl)$. The contradiction follows in the same way as in the previous lemma.

To get the result for f_l^u one can argue by contradiction. Using

$$|f_l^a(x)| = |f_l^u(x)| \prod_{j=0}^{n-1} |b(x+j\omega) b(x+(j+1)\omega)|^{1/2}$$

one can get that $|f_l^a(x)| \leq \exp(-Cl^2)$ for all x and this gives the same contradiction as before. \square

We recall for convenience some facts about stability of contracting and expanding directions of unimodular matrices. It follows from the polar decomposition that if $A \in SL(2, \mathbb{C})$ then there exist unit vectors $u_A^+ \perp u_A^-$ and $v_A^+ \perp v_A^-$ such that $Au_A^+ = \|A\| v_A^+$ and $Au_A^- = \|A\|^{-1} v_A^-$.

Lemma 4.4. ([GS08, Lemma 2.5]) *For any $A, B \in SL(2, \mathbb{C})$ we have*

$$\begin{aligned} |Bu_{AB}^- \wedge u_A^-| &\leq \|A\|^{-2} \|B\|, \quad |u_{BA}^- \wedge u_A^-| \leq \|A\|^{-2} \|B\|^2 \\ |v_{AB}^+ \wedge v_A^+| &\leq \|A\|^{-2} \|B\|^2, \quad |v_{BA}^+ \wedge Bv_A^+| \leq \|A\|^{-2} \|B\|. \end{aligned}$$

We will need the following estimate (cf. [GS08, (2.35)]) in the proof of Lemma 4.6.

Lemma 4.5. *If $A \in SL(2, \mathbb{C})$ and w_1, w_2 , and w_3 are unit vectors in the plane then*

$$|w_1 \wedge Aw_2| \leq |w_1 \wedge Aw_3| + \sqrt{2} \|A^{-1}\| |w_2 \wedge w_3|$$

and

$$|w_1 \wedge Aw_2| \leq |w_3 \wedge Aw_2| + \sqrt{2} \|A\| |w_1 \wedge w_3|$$

Proof. Since A preserves area we have

$$\begin{aligned} |w_1 \wedge Aw_2| &= |A^{-1}w_1 \wedge w_2| \leq |A^{-1}w_1 \wedge w_3| + \min |A^{-1}w_1 \wedge (w_2 \pm w_3)| \\ &\leq |w_1 \wedge Aw_3| + \|A^{-1}w_1\| \min \|w_2 \pm w_3\| \leq |w_1 \wedge Aw_3| + \|A^{-1}\| \sqrt{2} |w_2 \wedge w_3|. \end{aligned}$$

The second inequality follows from the first one. \square

Let \mathcal{G}_N be the set of points $x \in \mathbb{T}$ such that for any $1 \leq j \leq N$ and $|l| \leq 2N$ we have $|\log \|M_j^u(x + l\omega)\| - jL| \leq N^\sigma$, $\log \|M_j^u(x + l\omega)^{-1}\| \leq N^\sigma$, and $|\log |b(x + j\omega)| - D| \leq N^\sigma$. From Corollary 3.11, Corollary 3.2 and Theorem 2.1 we have that

$$\begin{aligned} \text{mes}(\mathbb{T} \setminus \mathcal{G}_N) &\leq (4N + 1) N \exp(-c_1 N^\sigma + r_N) + (4N + 1) N \exp(-c_2 N^\sigma) \\ &\quad + N \exp(-c_3 N^\sigma + r'_1) \leq \exp(-cN^\sigma) \end{aligned}$$

for N large enough. The choice of \mathcal{G}_N is such that all the estimates in the next lemma hold on this set.

Lemma 4.6. *Let $0 < \sigma < 1$. There exist constants $l_0 = l_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma, \sigma)$ and $N_0 = N_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma, \sigma)$ such that*

$$\begin{aligned} \text{mes}\{x \in \mathbb{T} : |f_N^u(x)| + |f_N^u(x + j_1\omega)| + |f_N^u(x + j_2\omega)| \leq \exp(NL_N - 100N^\sigma)\} \\ \leq \exp(-N^{\sigma/2}) \quad (4.7) \end{aligned}$$

for any $l_0 \leq j_1 \leq j_1 + l_0 \leq j_2 \leq N^{\sigma/8}$ and $N \geq N_0$.

Proof. Let $\{e_1, e_2\}$ be the standard basis of \mathbb{R}^2 . By (2.6) we have

$$\begin{aligned} f_N^u(x) &= M_N^u(x) e_1 \wedge e_2 \\ &= (M_N^u(x) [(u_N^+(x) \cdot e_1) u_N^+(x) + (u_N^-(x) \cdot e_1) u_N^-(x)]) \wedge e_2 \\ &= (u_N^+(x) \cdot e_1) \|M_N^u(x)\| v_N^+(x) \wedge e_2 + (u_N^-(x) \cdot e_1) \|M_N^u(x)\|^{-1} v_N^-(x) \wedge e_2. \end{aligned}$$

If $|f_N^u(x)| \leq \exp(NL_N - 100N^\sigma)$ then

$$\begin{aligned} \|M_N^u(x)\| |u_N^+(x) \cdot e_1| |v_N^+(x) \wedge e_2| - \|M_N^u(x)\|^{-1} |u_N^-(x) \cdot e_1| |v_N^-(x) \wedge e_2| \\ \leq \exp(NL_N - 100N^\sigma). \end{aligned}$$

From the above and the fact that $u_N^+(x) \cdot e_1 = u_N^-(x) \wedge e_1$ (recall that $u_N^+ \perp u_N^-$) one gets that on \mathcal{G}_N we have

$$\begin{aligned} |u_N^-(x) \wedge e_1| |v_N^+(x) \wedge e_2| &\leq \exp(N(L_N - L) - 99N^\sigma) + \exp(2N^\sigma - 2NL) \\ &\leq \exp(-90N^\sigma) \end{aligned}$$

and hence $|u_N^-(x) \wedge e_1| \leq \exp(-40N^\sigma)$ or $|v_N^+(x) \wedge e_2| \leq \exp(-40N^\sigma)$.

Suppose (4.7) fails. Then

$$\begin{aligned} \text{mes} \{x \in \mathcal{G}_N : |f_N^u(x)| + |f_N^u(x + j_1\omega)| + |f_N^u(x + j_2\omega)| \leq \exp(NL_N - 100N^\sigma)\} \\ > \exp(-N^{\sigma/2}) - \exp(-c_1N^\sigma) > \exp(-cN^{\sigma/2}). \end{aligned}$$

Let x be in the above set. By the preliminary discussion, either $|u_N^-(x) \wedge e_1| \leq \exp(-40N^\sigma)$ or $|v_N^+(x) \wedge e_2| \leq \exp(-40N^\sigma)$ has to hold for two of the points x , $x + j_1\omega$, $x + j_2\omega$.

We first assume that

$$|u_N^-(x + j_1\omega) \wedge e_1| \leq \exp(-40N^\sigma) \quad \text{and} \quad |u_N^-(x + j_2\omega) \wedge e_1| \leq \exp(-40N^\sigma). \quad (4.8)$$

We now compare $M_{j_2-j_1}^u(x + j_1\omega) u_N^-(x + j_1\omega)$ and $u_N^-(x + j_2\omega)$. From Lemma 4.5 it follows that

$$\begin{aligned} & |u_N^-(x + j_2\omega) \wedge M_{j_2-j_1}^u(x + j_1\omega) u_N^-(x + j_1\omega)| \\ & \leq |u_N^-(x + j_2\omega) \wedge M_{j_2-j_1}^u(x + j_1\omega) u_{N+j_2-j_1}^-(x + j_1\omega)| \\ & \quad + C \left\| M_{j_2-j_1}^u(x + j_1\omega)^{-1} \right\| |u_{N+j_2-j_1}^-(x + j_1\omega) \wedge u_N^-(x + j_1\omega)| \end{aligned}$$

Applying Lemma 4.4 with $A = M_N^u(x + j_2\omega)$ and $B = M_{j_2-j_1}^u(x + j_1\omega)$ for the first term, and $A = M_N^u(x + j_1\omega)$ and $B = M_{j_2-j_1}^u(x + (N + j_1)\omega)$ for the second term, yields

$$\begin{aligned} & |u_N^-(x + j_2\omega) \wedge M_{j_2-j_1}^u(x + j_1\omega) u_N^-(x + j_1\omega)| \\ & \leq \|M_N^u(x + j_2\omega)\|^{-2} \|M_{j_2-j_1}^u(x + j_1\omega)\| \\ & \quad + C \left\| M_{j_2-j_1}^u(x + j_1\omega)^{-1} \right\| \|M_N^u(x + j_1\omega)\|^{-2} \|M_{j_2-j_1}^u(x + (N + j_1)\omega)\|^2 \\ & \leq \exp((-2N + j_2 - j_1)L + 3N^\sigma) + C \exp((-2N + 2(j_2 - j_1))L + 5N^\sigma) \\ & \leq \exp(-NL) \quad (4.9) \end{aligned}$$

for $x \in \mathcal{G}_N$. Using Lemma 4.5, (4.8), and (4.9) we get

$$\begin{aligned} & |e_1 \wedge M_{j_2-j_1}^u(x + j_1\omega) e_1| \leq |e_1 \wedge M_{j_2-j_1}^u(x + j_1\omega) u_N^-(x + j_1\omega)| \\ & \quad + C \left\| M_{j_2-j_1}^u(x + j_1\omega)^{-1} \right\| |e_1 \wedge u_N^-(x + j_1\omega)| \\ & \leq |u_N^-(x + j_2\omega) \wedge M_{j_2-j_1}^u(x + j_1\omega) u_N^-(x + j_1\omega)| \\ & \quad + C \left\| M_{j_2-j_1}^u(x + j_1\omega) \right\| |e_1 \wedge u_N^-(x + j_2\omega)| \\ & \quad + C \left\| M_{j_2-j_1}^u(x + j_1\omega)^{-1} \right\| |e_1 \wedge u_N^-(x + j_1\omega)| \\ & \leq \exp(-NL) + C \exp((j_2 - j_1)L - 39N^\sigma) + C \exp(-39N^\sigma) \\ & \leq \exp(-30N^\sigma). \end{aligned}$$

On the other hand by (2.6) we have

$$|e_1 \wedge M_{j_2-j_1}^u(x+j_1\omega)e_1| = \left| \frac{b(x+j_2\omega)}{b(x+(j_2-1)\omega)} \right|^{1/2} |f_{j_2-j_1-1}^u(x+j_1\omega)|,$$

so

$$|f_{j_2-j_1-1}^u(x+j_1\omega)| \leq C \exp\left(\frac{1}{2}(N^\sigma - D) - 30N^\sigma\right) \leq \exp(-20N^\sigma).$$

The same type of estimate is obtained if we replace (j_1, j_2) in (4.8) with $(0, j_1)$ or $(0, j_2)$.

Now assume that

$$|v_N^+(x+j_1\omega) \wedge e_2| \leq \exp(-40N^\sigma) \quad \text{and} \quad |v_N^+(x+j_2\omega) \wedge e_2| \leq \exp(-40N^\sigma).$$

Similarly to the previous case (first use Lemma 4.5 and then Lemma 4.4) we have

$$\begin{aligned} & |v_N^+(x+j_2\omega) \wedge M_{j_2-j_1}^u(x+(N+j_1)\omega)v_N^+(x+j_1\omega)| \\ & \leq |v_{N+j_2-j_1}^+(x+j_1\omega) \wedge M_{j_2-j_1}^u(x+(N+j_1)\omega)v_N^+(x+j_1\omega)| \\ & + C \|M_{j_2-j_1}^u(x+(N+j_1)\omega)\| |v_N^+(x+j_2\omega) \wedge v_{N+j_2-j_1}^+(x+j_1\omega)| \\ & \leq \|M_N^u(x+j_1\omega)\|^{-2} \|M_{j_2-j_1}^u(x+(N+j_1)\omega)\| \\ & + C \|M_{j_2-j_1}^u(x+(N+j_1)\omega)\| \|M_N^u(x+j_2\omega)\|^{-2} \|M_{j_2-j_1}^u(x+j_1\omega)\|^2 \\ & \leq \exp((-2N+j_2-j_1)L+3N^\sigma) + C \exp((-2N+3(j_2-j_1))L+5N^\sigma) \\ & \leq \exp(-NL) \end{aligned}$$

for $x \in \mathcal{G}_N$ and

$$\begin{aligned} & |e_2 \wedge M_{j_2-j_1}^u(x+(N+j_1)\omega)e_2| \leq |e_2 \wedge M_{j_2-j_1}^u(x+(N+j_1)\omega)v_N^+(x+j_1\omega)| \\ & + C \|M_{j_2-j_1}^u(x+(N+j_1)\omega)^{-1}\| |e_2 \wedge v_N^+(x+j_1\omega)| \\ & \leq |v_N^+(x+j_2\omega) \wedge M_{j_2-j_1}^u(x+(N+j_1)\omega)v_N^+(x+j_1\omega)| \\ & + C \|M_{j_2-j_1}^u(x+(N+j_1)\omega)\| |e_2 \wedge v_N^+(x+j_2\omega)| \\ & + C \|M_{j_2-j_1}^u(x+(N+j_1)\omega)^{-1}\| |e_2 \wedge v_N^+(x+j_1\omega)| \\ & \leq \exp(-NL) + C \exp((j_2-j_1)L-39N^\sigma) + C \exp(-39N^\sigma) \leq \exp(-30N^\sigma). \end{aligned}$$

On the other hand by (2.6) we have

$$\begin{aligned} |e_2 \wedge M_{j_2-j_1}^u(x+(N+j_1)\omega)e_2| & = \left| \frac{b(x+(N+j_1)\omega)}{b(x+(N+j_1+1)\omega)} \right|^{1/2} \\ & \cdot |f_{j_2-j_1-1}^u(x+(N+j_1+1)\omega)|, \end{aligned}$$

so

$$|f_{j_2-j_1-1}^u(x + (N + j_1 + 1)\omega)| \leq C \exp\left(\frac{1}{2}(N^\sigma - D) - 30N^\sigma\right) \leq \exp(-20N^\sigma).$$

In conclusion

$$\text{mes}\{x \in \mathbb{T} : |f_l^u(x)| \leq \exp(-20N^\sigma)\} > \exp(-cN^{\sigma/2})$$

for some choice of l from $j_1 - 1, j_2 - 1, j_2 - j_1 - 1$. However, this contradicts the fact that Lemma 4.3 implies

$$\begin{aligned} \text{mes}\{x \in \mathbb{T} : |f_l^u(x)| \leq \exp(-20N^\sigma)\} &\leq \text{mes}\{x \in \mathbb{T} : |f_l^u(x)| \leq \exp(-N^\sigma)\} \\ &\leq \exp(-N^\sigma l^{-2}) \leq \exp(-N^{3\sigma/4}) < \exp(-cN^{\sigma/2}) \end{aligned}$$

(we used $l \leq N^{\sigma/8}$). \square

Lemma 4.7. *There exist constants $\kappa > 0$ and $N_0 = N_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ such that for $N \geq N_0$ we have*

$$\int_{\mathbb{T}} \frac{1}{N} |f_N^u(x)| dx > L_N - N^{-\kappa}.$$

Proof. Let Ω_N be the set of points $x \in \mathcal{G}_N$ such that

$$\begin{aligned} \min\{|f_N^u(x + j_1\omega)| + |f_N^u(x + j_2\omega)| + |f_N^u(x + j_3\omega)| \\ : 0 < j_1 < j_1 + l_0 \leq j_2 < j_2 + l_0 \leq j_3 \leq N^{\sigma/8}\} \\ > \exp(NL_N - 100N^\sigma), \end{aligned}$$

where l_0 is as in the previous lemma. If N is large enough then $\text{mes}(\mathbb{T} \setminus \Omega_N) \leq N \exp(-c_1 N^{\sigma/2}) < \exp(-cN^{\sigma/2})$.

Let $u(x) = \log |f_N^u(x)|/N$ and set $M = \lceil N^{\sigma/8}/l_0 \rceil$. For each $x \in \Omega_N$ we have that $|f_N^u(x + kl_0\omega)| > \exp(NL_N - 100N^\sigma)/3$ for all but at most two k 's, $1 \leq k \leq M$. We have

$$\begin{aligned} \langle u \rangle &:= \int_{\mathbb{T}} u(x) dx = \frac{1}{M} \sum_{k=1}^M \int_{\mathbb{T}} u(x + kl_0\omega) dx \\ &\geq \int_{\Omega_N} \left(\frac{M-2}{M} \left(L_N - 100N^{\sigma-1} - \frac{\log 3}{N} \right) + \frac{2}{M} \inf_{1 \leq k \leq M} u(x + kl_0\omega) \right) dx \\ &\quad + \frac{1}{M} \sum_{k=1}^M \int_{\mathbb{T} \setminus \Omega_N} u(x + kl_0\omega) dx. \quad (4.10) \end{aligned}$$

Let $v(x) = \log |f_N^a(x)|/N$. We have that

$$S := \sup_{z \in \mathcal{A}_{\rho_0''}} v(z) \leq \sup_{z \in \mathcal{A}_{\rho_0''}} \frac{1}{N} \log \|M_N^a(z)\| < \infty.$$

Let

$$v(z) = \int_{\mathcal{A}_{\rho'_0}} \log |z - \zeta| d\mu(\zeta) + h(z)$$

be the Riesz representation on $\mathcal{A}_{\rho'_0}$. Applying [GS08, Lemma 2.2] (see the proof of Lemma 3.4) we get that

$$\mu(\mathcal{A}_{\rho_0}) + \|h\|_{L^\infty(\mathcal{A}_{\rho_0})} \leq C \left(2S - \sup_{\mathbb{T}} v \right) \leq C(2S - \langle v \rangle). \quad (4.11)$$

Note that $\langle v \rangle$ is finite by subharmonicity. Since $\langle v \rangle = \langle u \rangle + D$, it follows that $\langle u \rangle$ is also finite. Using Cartan's estimate (see [GS01, Lemma 2.2]) we get that for any small $\epsilon > 0$ we have

$$\inf_{1 \leq k \leq M} v(x + kl_0\omega) \geq -C(2S - \langle v \rangle)N^\epsilon \quad (4.12)$$

up to a set not exceeding $CM \exp(-N^\epsilon)$ in measure. Since

$$u(x) = v(x) - \frac{1}{2N} (S_N(x) + S_N(x + \omega)) \quad (4.13)$$

we can use (4.12) and Theorem 2.1 to conclude that

$$\inf_{1 \leq k \leq M} u(x + kl_0\omega) > -C(2S - \langle u \rangle - D)N^\epsilon - D - N^\epsilon > (C\langle u \rangle - C')N^\epsilon$$

up to a set \mathcal{B}_N not exceeding $\exp(-cN^\epsilon)$ in measure. Therefore

$$\begin{aligned} \int_{\Omega_N} \inf_{1 \leq k \leq M} u(x + kl_0\omega) dx &> (C\langle u \rangle - C')N^\epsilon + \int_{\Omega_N \cap \mathcal{B}_N} \inf_{1 \leq k \leq M} u(x + kl_0\omega) \\ &> (C\langle u \rangle - C')N^\epsilon - \sum_{k=1}^M \int_{\Omega_N \cap \mathcal{B}_N} |u(x + kl_0\omega)| dx. \end{aligned}$$

Now (4.10) becomes

$$\begin{aligned} \langle u \rangle &\geq \left(1 - \frac{2}{M}\right) \left(L_N - 100N^{\sigma-1} - \frac{\log 3}{N}\right) + \frac{(C\langle u \rangle - C')N^\epsilon}{M} \\ &\quad - \frac{2}{M} \sum_{k=1}^M \int_{\Omega_N^c \cup \mathcal{B}_N} |u(x + kl_0\omega)|. \end{aligned}$$

Using Lemma 4.3 (with $\sigma = 3$) and reasoning as in the proof of Lemma 3.8 we get that $\|u\|_{L^2(\mathbb{T})} \leq CN^3$ and consequently

$$\int_{\Omega_N^c \cup \mathcal{B}_N} |u(x + kl_0\omega)| dx \leq (\text{mes}\{\Omega_N^c \cup \mathcal{B}_N\})^{1/2} \|u\|_{L^2(\mathbb{T})} \leq CN^3 \exp(-cN^\epsilon).$$

Now it is straightforward to reach the conclusion. \square

Corollary 4.8. *Let*

$$\frac{1}{n} \log \|f_n^a(z)\| = \int_{\mathcal{A}_{\rho'_0}} \log |z - \zeta| d\mu_n(\zeta) + h_n(z)$$

be the Riesz representation on $\mathcal{A}_{\rho'_0}$. There exists a constant $C_0 = C_0(\|a\|_\infty, I_{a,E}, \|b\|_, |E|, \omega, \gamma, \rho_0, \rho'_0, \rho''_0)$ such that*

$$\mu_n(\mathcal{A}_{\rho_0}) + \|h_n\|_{L^\infty(\mathcal{A}_{\rho_0})} \leq C_0.$$

Proof. It suffices to obtain the bound for large n . The bound follows from (4.11) and the previous lemma. \square

Lemma 4.9. *There exist constants $\sigma_0 > 0$, $c_0 = c_0(I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$, and $C_0 = C_0(I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ such that for every integer n and any $\delta > 0$ we have*

$$\text{mes}\{x \in \mathbb{T} : |\log |f_n^a(x)| - \langle \log |f_n^a| \rangle| > n\delta\} \leq C_0 \exp(-c_0 \delta n^{\sigma_0}).$$

The same estimate with possibly different c_0 and C_0 holds for f_n^u .

Proof. It is enough to establish the estimate for n large enough. Let $u(x) = \log |f_n^u(x)|/n$ and $v(x) = \log |f_n^a(x)|/n$. By the previous lemma (recall that $\langle v \rangle = \langle u \rangle + D$) and Proposition 3.14 we have that there exists a small $\kappa > 0$ such that

$$\begin{cases} \langle v \rangle \geq L_n^a - n^{-\kappa} \\ \sup_{\mathbb{T}} v \leq L_n^a + n^{-\kappa}. \end{cases}$$

This implies that

$$\|v - \langle v \rangle\|_{L^1(\mathbb{T})} \leq C n^{-\kappa}$$

and hence by [BGS01, Lemma 2.3] we have

$$\|v\|_{BMO(\mathbb{T})} = \|v - \langle v \rangle\|_{BMO(\mathbb{T})} \leq C \|v - \langle v \rangle\|_{L^1(\mathbb{T})}^{1/2} \leq C n^{-\kappa/2}.$$

As in the proof of [GS08, Proposition 2.11] we note that in order to get the conclusion of [BGS01, Lemma 2.3] we just need the bounds on the Riesz representation of v . By the John-Nirenberg inequality we get

$$\text{mes}\{x \in \mathbb{T} : |v(x) - \langle v \rangle| > \delta\} \leq C \exp(-c\delta n^{\kappa/2}).$$

Using (4.13) we have

$$\begin{aligned} \text{mes}\{x \in \mathbb{T} : |u(x) - \langle u \rangle| > \delta\} &\leq \text{mes}\left\{x \in \mathbb{T} : |v(x) - \langle v \rangle| > \frac{\delta}{2}\right\} \\ &\quad + \text{mes}\left\{x \in \mathbb{T} : \left|\frac{1}{2n}(S_n(x) + S_n(x + \omega)) - D\right| > \frac{\delta}{2}\right\} \\ &\leq C \exp(-c\delta n^{\kappa/2}/2) + 2 \exp(-c'\delta n/2 + r_n) \leq C' \exp(-c''\delta n^{\kappa/2}/2). \end{aligned}$$

This concludes the proof. \square

Next we will use the Avalanche Principle to refine the previous estimate.

Proposition 4.10. *There exist constants $c_0 = c_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$, $C_0 = C_0(\omega) > \alpha + 2$, and $C_1 = C_1(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ such that for every integer $n > 1$ and any $\delta > 0$ we have*

$$\text{mes}\{x \in \mathbb{T} : |\log |f_n^a(x)| - \langle \log |f_n^a| \rangle| > n\delta\} \leq C_1 \exp\left(-c_0\delta n (\log n)^{-C_0}\right).$$

Proof. It is enough to establish the estimate for n large enough. We have that

$$\begin{bmatrix} f_n^u(x) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} M_n^u(x) \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} =: \mathcal{M}_n^u(x).$$

We define \mathcal{M}_n^a analogously. We obviously have that $|f_n^a(x)| = \|\mathcal{M}_n^a(x)\|$.

Let $l = \lceil (\log n)^{2/\sigma_0} \rceil$ with σ_0 as in Lemma 4.9. Let $n = l + (m-2)l + l'$ with $2l \leq l' \leq 3l$. We want to apply the Avalanche Principle to $\mathcal{M}_n^u(x) = \prod_{j=m}^1 A_j^u(x)$ where $A_j^u(x) = M_l^u(x + (j-1)l\omega)$, $j = 2, \dots, m-1$,

$$A_1^u(x) = M_l^u(x) \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f_l^u(x) & 0 \\ \star & 0 \end{bmatrix},$$

and

$$A_m^u(x) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} M_{l'}^u(x) = \begin{bmatrix} f_{l'}^u(x) & \star \\ 0 & 0 \end{bmatrix}.$$

We define the matrices A_j^a analogously. We clearly have that

$$\log |f_l^u(x)| \leq \log \|A_1^u(x)\| \leq \log \|M_l^u(x)\|,$$

and an analogous estimate for $\log \|A_m^u\|$. Now it follows from Corollary 3.11, Lemma 4.9, and Lemma 4.7 that the hypotheses of Lemma 3.7 are satisfied and hence

$$\log \|\mathcal{M}_n^a(x)\| + \sum_{j=2}^{m-1} \log \|A_j^a(x)\| - \sum_{j=1}^{m-1} \log \|A_{j+1}^a(x) A_j^a(x)\| = O\left(\frac{1}{l}\right)$$

up to a set of measure less than $3n \exp(-cl^{\sigma_0}) < \exp(-c'(\log n)^2)$. Note that, as before, we checked the conditions of the Avalanche Principle for \mathcal{M}_n^u , but we wrote the conclusion for \mathcal{M}_n^a . By letting

$$u_0(x) = \log \|A_m^a(x) A_{m-1}^a(x)\| + \log \|A_2^a(x) A_1^a(x)\|$$

we rewrite the previous relation as

$$\begin{aligned} \log \|\mathcal{M}_n^a(x)\| + \sum_{j=2}^{m-1} \log \|M_l^a(x + (j-1)l\omega)\| \\ - \sum_{j=2}^{m-2} \log \|M_{2l}^a(x + (j-1)l\omega)\| - u_0(x) = O\left(\frac{1}{l}\right). \end{aligned}$$

Note that

$$\begin{aligned} \log \|f_{l+l'}^a(x + (m-2)l\omega)\| + \log \|f_{2l}^a(x)\| &\leq u_0(x) \\ &\leq \log \|M_{l+l'}^a(x + (m-2)l\omega)\| + \log \|M_{2l}^a(x)\|. \end{aligned} \quad (4.14)$$

We apply the Avalanche Principle $l-1$ more times. At each step we decrease the length of A_m by one and increase the length of A_1 by one. Adding the resulting estimates and dividing by l yields

$$\begin{aligned} \log \|\mathcal{M}_n^a(x)\| + \sum_{j=l}^{(m-1)l-1} \frac{1}{l} \log \|M_l^a(x + j\omega)\| - \sum_{j=l}^{(m-2)l-1} \frac{1}{l} \log \|M_{2l}^a(x + j\omega)\| \\ - \sum_{k=0}^{l-1} \frac{1}{l} u_k(x) = O\left(\frac{1}{l}\right) \end{aligned} \quad (4.15)$$

up to a set of measure less than $l \exp(-c(\log n)^2) < \exp(-c'(\log n)^2)$. The functions u_k , $k = 1, \dots, l-1$ are defined analogously to u_0 and satisfy estimates analogous to (4.14). Based on these estimates it is straightforward to conclude (see Lemma 3.4 and (4.11)) that there is an uniform bound for the Riesz representations of u_k/l , $k = 1, \dots, l-1$. Hence we can use Theorem 2.1 to get

$$\sum_{k=0}^{l-1} \frac{1}{l} u_k(x) - \sum_{k=0}^{l-1} \frac{1}{l} \langle u_k \rangle = O(l(\log n)^2) = O((\log n)^{2+2/\sigma_0})$$

up to a set of measure less than $l \exp(-c(\log n)^2) < \exp(-c'(\log n)^2)$. On the other hand, using Theorem 3.10 we have

$$\begin{aligned} \sum_{j=l}^{(m-1)l-1} \frac{1}{l} \log \|M_l^a(x + j\omega)\| - \sum_{j=l}^{(m-2)l-1} \frac{1}{l} \log \|M_{2l}^a(x + j\omega)\| \\ = (m-2)lL_l^a - (m-3)lL_{2l}^a + O((\log n)^p) \end{aligned}$$

up to a set of measure less than $\exp(-c(\log n)^p)$. We can now conclude from (4.15) that

$$\log |f_n^a(x)| + (m-2)lL_l^a - (m-3)lL_{2l}^a - \sum_{k=0}^{l-1} \frac{1}{l} \langle u_k \rangle = O((\log n)^{C_2})$$

up to a set of measure less than $\exp(-c(\log n)^2)$, where $C_2 = \max\{p, 2 + 2/\sigma_0\}$. Integrating the above relation and then subtracting it, yields

$$|\log |f_n^a(x)| - \langle \log |f_n^a| \rangle| \leq C(\log n)^{C_2} \quad (4.16)$$

up to a set of measure less than $\exp(-c(\log n)^2)$. Note that the exceptional set was handled by using the fact that $\|\log |f_n^a|\|_{L^2(\mathbb{T})} \leq Cn$. This follows from

$$\|\log |f_n^a| - \langle \log |f_n^a| \rangle\|_{L^2(\mathbb{T})} \leq Cn \quad (4.17)$$

and $|\langle \log |f_n^a| \rangle| \leq Cn$. The first estimate is an immediate consequence of Lemma 4.9 and Lemma 3.8. The second estimate can be deduced from Lemma 4.7.

Let \mathcal{B} be the exceptional set for (4.16). Let

$$\log |f_n^a| - \langle \log |f_n^a| \rangle = u_0 + u_1$$

where $u_0 = 0$ on \mathcal{B} and $u_1 = 0$ on $\mathbb{T} \setminus \mathcal{B}$. By (4.16) and (4.17) we have that $\|u_0 - \langle u_0 \rangle\|_{L^\infty(\mathbb{T})} \leq C(\log n)^{C_2}$ and

$$\|u_1\|_{L^2(\mathbb{T})} \leq Cn\sqrt{\text{mes}(\mathcal{B})} \leq \exp\left(-c(\log n)^2\right).$$

Applying [BGS01, Lemma 2.3] we have

$$\|\log |f_n^a|\|_{BMO(\mathbb{T})} \leq C \left((\log n)^{C_2+2} + \sqrt{n \exp\left(-c(\log n)^2\right)} \right) \leq C' (\log n)^{C_0}.$$

The conclusion follows from the John-Nirenberg inequality. \square

Lemma 4.11. *There exists a constant $C_0 = C_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ such that*

$$|\langle \log |f_n^a| \rangle - nL_n^a| \leq C_0$$

for all integers.

Proof. Subtracting the Avalanche Principle expansions for \mathcal{M}_n^a and M_n^a at scale $l \approx (\log n)^A$ and then integrating, yields

$$|\langle \log |f_n^a| \rangle - nL_n^a| \leq CR \left(4(\log n)^A \right) + O\left(\frac{1}{l}\right)$$

where

$$R(n) = \sup_{n/2 \leq m \leq n} |\langle \log |f_m^a| \rangle - mL_m^a|.$$

Iterating this estimate yields the desired conclusion (cf. [GS08, Lemma 3.5]). \square

We now prepare to prove the estimate on the number of eigenvalues. Fix $E_0 \in \mathbb{R}$ such that $L(E_0) \geq \gamma > 0$. As a consequence of Corollary 3.17 and Lemma 3.9 it follows that there exists a disk \mathcal{D} around E_0 such that $L(E) \geq \gamma/2$ on I . In what follows we also fix \mathcal{D} . Note that the existence of the disk \mathcal{D} would follow from the continuity of the Lyapunov exponent, which is known from [JM11]. However, we also need the information on the modulus of continuity provided by Corollary 3.17. This information follows from the Hölder continuity of the Lyapunov exponent proved in [Tao11], but we use Corollary 3.17 in order to keep the paper self-contained. The following deviations estimate in E will be needed in the proof of the estimate.

Lemma 4.12. *Let C_0 be as in Proposition 4.10. There exist constants $c_0 = c_0(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ and $C_1 = C_1(\|a\|_\infty, I_{a,E}, \|b\|_*, |E|, \omega, \gamma)$ such that for every integer $n > 1$ and every $\delta \geq \delta_0$ there exists a set $\mathcal{B}_{n,\delta} \subset \mathbb{T}$ with $\text{mes } \mathcal{B}_{n,\delta} < C_1 \exp(-c_0 \delta (\log n)^{-C_0})$, such that for each $x \in \mathbb{T} \setminus \mathcal{B}_{n,\delta}$ there exists $\mathcal{E}_{n,\delta,x} \subset \mathcal{D}$, with $\text{mes } \mathcal{E}_{n,\delta,x} < C_1 \exp(-c_0 \delta (\log n)^{-C_0})$, such that*

$$|\log |f_n^a(x, E)| - nL_n^a(E)| \leq \delta, \quad (4.18)$$

for any $E \in \mathcal{D} \setminus \mathcal{E}_{n,\delta,x}$.

Proof. From Proposition 4.10 and Lemma 4.11 it follows that (4.18) holds for $\delta \geq \delta_0$, and $(x, E) \in \mathbb{T} \times \mathcal{D}$ except for a set of measure $C \exp(-c\delta/(\log n)^{C_0})$. The conclusion follows by Fubini's Theorem and Chebyshev's inequality. \square

Theorem 4.13. *Let $C_0 = C_0(\omega)$ be as in Proposition 4.10. There exist constants $C_1 = C_1(\|a\|_\infty, \|b\|_*, |E_0|, \omega, \gamma)$, $C_2 = C_2(\|a\|_\infty, I_{a,\mathcal{D}}, \|b\|_*, |E_0|, \omega, \gamma)$, and $n_0 = n_0(\|a\|_\infty, I_{a,\mathcal{D}}, \|b\|_*, |E_0|, \omega, \gamma)$ such that for any $x_0 \in \mathbb{T}$, $E_0 \in \mathbb{R}$, and $n \geq n_0$ one has*

$$\#\{E \in \mathbb{R} : f_n^a(x_0, E) = 0, |E - E_0| < n^{-C_1}\} \leq C_2 (\log n)^{2C_0}$$

and

$$\#\{z \in \mathbb{C} : f_n^a(z, E_0) = 0, |z - x_0| < n^{-1}\} \leq C_2 (\log n)^{2C_0}.$$

Proof. From (4.18) it follows that there exist x_1, E_1 such that

$$|x_1 - x_0| \leq C \exp(-c(\log n)^{C_0}),$$

$$|E_1 - E_0| \leq C \exp(-c(\log n)^{C_0}),$$

and

$$\log |f_n^a(x_1, E_1)| \geq nL_n(E_1) - (\log n)^{2C_0}. \quad (4.19)$$

Let $R = n^{-2C_3}$, where C_3 is the constant C_1 from Corollary 3.17, and let $\nu_{x,E}(r) = \#\{E : f_n^a(x, E') = 0, |E' - E| \leq r\}$. Using Jensen's formula we have that

$$\begin{aligned} \nu_{x_1, E_1}(3R) &\leq C \int_0^{4R} \frac{\nu_{x_1, E_1}(t)}{t} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log |f_n^a(x_1, E_1 + 4Re^{i\theta})| d\theta - \log |f_n^a(x_1, E_1)| \end{aligned} \quad (4.20)$$

By Proposition 3.14 we have

$$\log |f_n^a(x_1, E)| \leq nL_n^a(E) + C(\log n)^p$$

for $E \in \mathcal{D}$. Using this, together with (4.19) and (4.20) yields

$$\nu_{x_1, E_1}(3R) \leq C \left(\sup_{|E - E_1| = 4R} (n(L_n^a(E) - L_n^a(E_1))) + (\log n)^{2C_0} \right).$$

For E such that $|E - E_0| \leq R$ we have that $|E - E_1| \leq n^{-C_3}$ and hence by Corollary 3.17 we have that $|n(L_n^a(E) - L_n^a(E_1))| \leq n^{-C}$. We can now conclude that

$$\nu_{x_1, E_0}(2R) \leq \nu_{x_1, E_1}(3R) \leq C(\log n)^{2C_0}. \quad (4.21)$$

Using the Mean Value Theorem we can conclude that

$$\|H^{(n)}(x_0) - H^{(n)}(x_1)\| \leq C|x_0 - x_1| \leq C \exp(-c(\log n)^{C_0}).$$

Let $E_j^{(n)}(x)$, $j = 1, \dots, n$ be the eigenvalues of $H^{(n)}(x)$ ordered increasingly. Since $H^{(n)}(x_0)$ and $H^{(n)}(x_1)$ are Hermitian it follows that

$$|E_j^{(n)}(x_0) - E_j^{(n)}(x_1)| \leq C \exp(-c(\log n)^{C_0}).$$

This implies that $\nu_{x_0, E_0}(R) \leq \nu_{x_1, E_0}(2R)$ and now the first estimate follows from (4.21).

The second estimate follows in a similar way. From Proposition 4.10 it follows that there exists x_1 such that $|x_1 - x_0| \leq C \exp(-c(\log n)^{C_0})$ and

$$\log |f_n^a(x_1, E_0)| \geq nL_n(E_0) - (\log n)^{2C_0}. \quad (4.22)$$

Let $\nu_x(r) = \#\{z \in \mathbb{C} : f_n^a(z, E_0) = 0, |z - x| < r\}$. Using Jensen's formula, (4.22) and Proposition 3.14, as before, yields

$$\begin{aligned} \nu_{x_0}(n^{-1}) &\leq \nu_{x_1}(2n^{-1}) \\ &\leq C \left(\sup_{r \in (1-3n^{-1}, 1+3n^{-1})} (n(L_n^a(r, E_0) - L_n^a(1, E_0))) + (\log n)^{2C_0} \right) \\ &\leq C' (\log n)^{2C_0}. \end{aligned}$$

For the last inequality we used Corollary 3.13. This concludes the proof. \square

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